An Axiomatic Characterization of
the Theil Measure of Income Inequality

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This paper provides a characterization of a frequently used measure of income inequality. It has been known for some time that the Theil measure of income inequality (1) is consistent with the Lorenz criterion, when it applies, and (2) exhibits a simple and empirically useful decomposition by population subgroup into within-group and between-group terms. The major theorem establishes the converse: the decomposability property defines the Theil measure uniquely (up to a positive multiple) among all Lorenz-consistent measures. *Journal of Economic Literature* Classification Number: 024.

1. INTRODUCTION

Measures of income inequality are often enlisted in public finance and development studies to assess the existing distribution of income in a given region, or to evaluate the distributional effects of a particular policy. The typical questions addressed in these studies are (i) How has inequality in region A changed over time? (ii) Is inequality in region B greater than that in region C? (iii) What are the major sources of inequality in region D?

Atkinson [2] and Sen [24] have stressed that the answers to these questions may depend crucially on the choice of inequality measure, and suggest basic properties for measures to satisfy. Primary among these is the requirement that inequality should increase under a transfer from poor to rich, a property that Sen calls the Pigou–Dalton transfers principle after the work of Pigou [22] and Dalton [10]. The intuitive appeal of the Pigou–Dalton criterion is further supported by the results of Atkinson [2],

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Dasgupta *et al.* [11] and Rothschild and Stiglitz [23], linking it with the well-known Lorenz criterion and with the social welfare approach to inequality.

In addition to this central property of inequality measures, Atkinson suggests that a measure should be symmetric and homogeneous of degree zero in all incomes, filling out what Fields and Fei [14] call the three "basic axioms" for inequality measurement. While several common measures of dispersion fail to satisfy the three—for instance, the variance and the standard deviation of logarithms—Fields and Fei note that there remains quite a range of measures consistent with all three. They conclude that "A feasible and desirable direction for future research is to investigate what further axioms could be introduced to complete the axiomatic system..." [14, p. 315]. The present study may be regarded as a step in this direction.

In this paper it is shown that the addition of a simple decomposability property (see condition TD, below) to the three basic properties implies that the measure is the Theil measure of inequality (up to a positive multiple). The essential justification for using the Theil measure over any other "reasonable" measure rests in the usefulness or desirability of this additional property.

Dalton's population principle (see condition PP, below) is often proposed as a fourth property for inequality measures to satisfy. A second theorem shows that an alternative decomposability property is sufficient to single out the Theil measure from among all measures satisfying the four. Both theorems follow from a general uniqueness theorem for the Theil measure, which we prove with the aid of a result in information theory by Lee [20]. In addition we are able to generalize a recent result by Bourguignon [5], and remove his objectionable differentiability assumption.

2. NOTATION

Throughout the paper, i, j, k, l, m, n denote integers, $\alpha, \beta, \lambda, r, s, t$, are real numbers, and v, w, x, y, z are vectors. The symbol $\mathbb{R}$ represents the set of all real numbers, and for $n \geqslant 1$, $\mathbb{R}^n$ denotes the set of all vectors having n real entries. Three vectors in $\mathbb{R}^n$ are singled-out and denoted by special symbols: the origin is $0^n = (0,..., 0)$; the vector of ones is $u^n = (1,..., 1)$; and the first usual basis vector is $e^n = (1, 0,..., 0)$.

For x and y in $\mathbb{R}^n$, the expression $x \geqslant y$ is defined by $x_i \geqslant y_i$ for $i = 1,..., n$; $x > y$ denotes $x \geqslant y$ and $x \neq y$; and $x \gg y$ indicates that $x_i > y_i$ for $i = 1,..., n$. Furthermore, for $x \geqslant 0^n$, the sum of all coordinates $\sum_{i=1}^{n} x_i$ is denoted by the symbol $|x|$, and $\bar{x} \equiv |x|/n$ denotes the mean. The notation $(x; z)$ is used to indicate the "combination" vector $(x_1,..., x_n, z_1,..., z_m)$, where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$. When a given vector y is partitioned into $l \geqslant 2$

smaller vectors, the subvectors are denoted by parenthetical superscripts; i.e., $y = (y^{(1)}; \ldots; y^{(l)})$.

3. INEQUALITY MEASURES AND PROPERTIES

How does one define an inequality measure? Here we adopt a general definition that allows both population size and total income to be variable. First, for a given population size $n \geqslant 1$, consider the set $D_n = \{x \in \mathbb{R}^n : x > 0^n\}$ of *n-distributions*. Each *n*-distribution x fixes a particular scheme from among the infinite possible ways of distributing a total amount of income $|x|$ among n persons. We use the term *inequality index* to indicate a function $I^n : D_n \to \mathbb{R}$ which gives comparisons of inequality at a given level of population.[1] In contrast, an *inequality measure* I offers comparisons *across* population sizes as well, and is defined on $\mathcal{D} \equiv \bigcup_{n=1}^{\infty} D_n$, the set of all *distributions*. Clearly, each inequality measure $I : \mathcal{D} \to \mathbb{R}$ is uniquely associated with a family of indices $\{I^n\}$, one for each population level.[2]

The class of all possible inequality measures is rather large, and contains many functions quite clearly unrelated to commonly held notions of inequality. A number of properties have come to be accepted as "basic" properties of measures of inequality, and as such serve to reduce the number of allowable measures. Dalton [10] was first to propose this "axiomatic" approach, so it is quite appropriate that the central property of inequality measurement bears his name. In words, the Pigou–Dalton transfers principle requires inequality to increase when income is transferred from one person to anyone richer (or no less rich). Following Rothschild and Stiglitz [23], we say that y is obtained from x by a *regressive transfer* if, for some i and j: (i) $x_i \leqslant x_j$, (ii) $y_j - x_j = x_i - y_i > 0$, and (iii) $y_k = x_k$ for all $k \neq i, j$. Hence, x and y are identical but for a positive transfer of income from the "poorer" person i to the "richer" person j.

PIGOU–DALTON TRANSFERS PRINCIPLE (PD). If $y \in \mathcal{D}$ is obtained from $x \in \mathcal{D}$ by a regressive transfer, then $I(y) > I(x)$.

Thus, condition PD implies that a regressive transfer increases inequality.

A second property ensures that the identities of income receivers are not relevant to the question of income inequality. We say that y is obtained from x by a *permutation of incomes* if $y = xP$ for some permutation matrix P of order n.[3]

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[1] Some formulas admit the possibility of *infinite* inequality at some distribution, e.g., see [26, p. 126]. In this case, one would have to use the extended reals $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ as the range.

[2] This usage corresponds with Donaldson and Weymark [12], although instead of considering the function I, they consider the family of indices $\{I^n\}$.

[3] A permutation matrix is a square matrix each of whose rows and columns contains a

SYMMETRY (S). If $y \in \mathscr{D}$ is obtained from $x \in \mathscr{D}$ by a permutation of incomes, then $I(y) = I(x)$.

Thus, condition S implies that inequality remains unchanged when individuals trade places.

The third property usually adopted is homogeneity of degree zero in all incomes: so long as the proportional shares of income are maintained, a change in the total income leaves inequality fixed. We say that a distribution y is obtained from another distribution x by a *proportionate change in incomes* if $y = \alpha x$ for some $\alpha > 0$.

HOMOGENEITY (H). If $y \in \mathscr{D}$ is obtained from $x \in \mathscr{D}$ by a proportionate change in incomes, then $I(y) = I(x)$.

Condition H says that a proportionate change in incomes leaves inequality unchanged.[4]

Fields and Fei [14] take PD, S, and H as the defining characteristics of "relative" measures of inequality, and stress that the three leave much room for choice among existing and possible measures of inequality. In fact, since each property is a restriction on I^n in isolation without reference to cross-population comparisons, PD, S, and H admit even more measures than Fields and Fei indicate. Consider the measure that takes the Gini coefficient index at even sized populations, and the coefficient of variation index at odd. This absurd measure quite clearly satisfies all three properties.

The above example serves to point out the desirability of a property which would coordinate the indices into one cohesive measure. Another property suggested by Dalton [10] does this in a particularly natural way. We say that y is a *replication* of x if $y, x \in \mathscr{D}$ and for some $m \geqslant 2$ we have $y = (y^{(1)}; ...; y^{(m)})$, where each $y^{(i)} = x$.

POPULATION PRINCIPLE (PP). If y is a replication of x, then $I(y) = I(x)$.

Thus, under condition PP a distribution and its replications have the same level of inequality.

In addition to assuring a degree of conformity among the indices, the population principle acts as a "population homogeneity" property: given that the proportion of the population receiving each income is fixed, changing (e.g., doubling) the population size has no effect on the measured level of inequality. This property leads us to view inequality in *per capita* terms.

Finally, a measure is *nontrivial* if $I(x) \neq I(y)$ for some $x, y \in \mathscr{D}$, it is

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single "one" and the rest "zeros." The effect of multiplying a vector $(y_1, ..., y_n)$ by such a matrix is to reorder the components.

[4] Sen [24] and Kolm [19] discuss condition H in greater detail.

directed (D) if for each $n \geqslant 1$, $I(y) \geqslant I(\bar{y}u^n)$ for all $y \in D_n$, and it is *normalized* (N) if $I(\beta u^n) = 0$ for all $n \geqslant 1$ and $\beta > 0$.[5]

4. The Theil Measure and Theil Decomposability

In 1967, Theil [26] introduced an inequality measure $T: \mathscr{D} \to \mathbb{R}$ whose indices are given by

$$T^n(y) = \sum_{i=1}^{n} \frac{y_i}{|y|} \ln \left(n \frac{y_i}{|y|} \right), \qquad y \in D_n, \tag{1}$$

for each $n \geqslant 1$, where $0 \ln(0)$ is always taken to be 0.[6] It is clear that each index T^n is largest at $T^n(e^n) = \ln(n)$; potential inequality increases unboundedly with the population size. Note that T is nontrivial, normalized, and directed. Many authors have pointed out that T satisfies PD, S, H, and PP.

Theil [26, 27] has shown that his measure has an additional, useful property. In a population consisting of several groups of income-receivers, inequality may be expressed as the sum of two distinct terms: (i) "within-group" inequality, as represented by the sum of the inequality levels of each group weighted by the share of income it receives as a proportion of the total income of the population; and (ii) "between-group" inequality, given by an application of the measure to a "smoothed" income distribution wherein each income of a group is replaced by the mean income of the group [26, p. 95]. For example, if $y = (1, 3, 5, 7, 3)$ is broken down into $y^{(1)} = (1, 3)$ and $y^{(2)} = (5, 7, 3)$, then the decomposability property implies that $I(y) = (4/19) I(y^{(1)}) + (15/19) I(y^{(2)}) + I(2, 2, 5, 5, 5)$. In precise terms this may be formulated as follows:

THEIL DECOMPOSABILITY (TD).   For any $m \geqslant 2$, any $n_1, ..., n_m \geqslant 1$, and any distributions $y^{(i)} \in D_{n_i}$ $(i = 1, ..., m)$,

$$I(y) = \sum_{i=1}^{n} \frac{|y^{(i)}|}{|y|} I(y^{(i)}) + I(\bar{y}^{(1)}u^{n_1}; \bar{y}^{(2)}u^{n_2}; ...; \bar{y}^{(m)}u^{n_m}),$$

$$\text{where} \quad y = (y^{(1)}; ...; y^{(m)}). \tag{2}$$

Due to its additive form, the simple weights in the within-group term, and

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[5] Assumptions N and D are often combined into an assumption of nonnegative values for the inequality measure, with equality at zero; see, for instance, Kolm [19]. Nontriviality, N, and D are often assumed simultaneously by stating that the nonnegative valued measure is to assume the value zero exactly at points of complete equality; e.g., Shorrocks [25].

[6] Note that $\alpha \ln(n\alpha)$ tends to zero as $\alpha > 0$ approaches zero, which assures that T is continuous on D_n. See [15] for details. As usual, $\ln(\cdot)$ refers to the natural logarithm.

the easily interpreted between-group term, Theil decomposability has received widespread use in studies of income inequality, beginning with Theil's [27] own analysis of inequality among the states of the United States.[7]

Note that TD requires the income-share decomposition (2) to hold for any (ordered) partition of a given distribution y into group distributions $y^{(1)},..., y^{(m)}$ in $\mathscr{D}$.[8] One may obtain less stringent properties by reducing the set of group distributions over which the decomposition is required to hold. For instance, the population principle can easily be interpreted as an income-share weighted decomposability property that applies only to m-replications of identical n-distributions, $(y^{(1)};...; y^{(m)}) = (z;...; z)$. The within-group inequality term is $\sum (1/m) I(y^{(i)})$ where $1/m$ is clearly the income share of $y^{(i)}$, while the between-group term is nonexistent since each of the component distributions is identical.

Alternatively, one may wish to consider a decomposability property that also applies only when each subgroup has the same population size, but allows the group distributions to be different.

WEAK THEIL DECOMPOSABILITY (WD). For all $m \geqslant 2$, $n \geqslant 1$, and any $y^{(1)}, y^{(2)},..., y^{(m)} \in D_n$,

$$I(y) = \sum_{i=1}^{m} \frac{|y^{(i)}|}{|y|} I(y^{(i)}) + I(\bar{y}^{(1)}, \bar{y}^{(2)},..., \bar{y}^{(m)}),$$

$$\text{where} \quad y = (y^{(1)}; y^{(2)};...; y^{(m)}). \quad (3)$$

In contrast to Theil decomposability, weak Theil decomposability applies only to partitions yielding equal-sized groups. Further, the between-group term in WD is of a much simpler form: the measure is applied to the m-distribution of group means rather than to the larger "smoothed" distribution used in TD. Nonetheless, since both properties incorporate an income-share weighted decomposition, WD is closely linked with TD. The next three results give the relationships among conditions TD, WD, PP, and N.

PROPOSITION 1. *Let an inequality measure I satisfy TD. Then I must also satisfy PP and N.*

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[7] See [13] for a discussion of the applications of TD. Interesting comparisons of TD with other forms of decomposition may be found in [5, 9, 25]. An alternative approach to decomposition analysis via social welfare considerations is taken in [4].

[8] Requiring the group distributions to lie in $\mathscr{D}$ rules out the possibility of group distributions that are vectors of zeros. Note also that our characterization of TD covers only ordered partitions of the vector y, and not arbitrary partitions of the set of incomes in y. Of course, TD and S together imply the latter property.

Proof. If I satisfies TD, then where $n \geqslant 1$, $\beta > 0$, and $y = \beta u^{2n}$, we have $I(y) = I(\beta u^n) + I(y)$, which yields N. Now let $m \geqslant 2$, $n \geqslant 1$, and suppose $z \in D_n$. Then where y is an m-replication of z, we have $I(y) = I(z) + I(\bar{z} u^{nm})$, where the final term vanishes by N. Hence, PP holds.

Theil decomposability, then, subsumes the population principle and normalization. One interesting implication of this result is that if I satisfies TD, then where l, m, and n are positive integers such that $l + n = m$, and $y = (x; \bar{x} u^l)$ for some $x \in D_n$, we must have $I(y) = (n/m) I(x)$. In words, if a population is enlarged from n to m persons by adding l many persons with the mean income, then inequality must decrease by a factor of l/m. In particular, if we keep on adding persons with the mean income until the population size doubles, the resulting distribution will have *half* the inequality of the original distribution.[9]

The next result shows that WD implies neither N nor PP, but renders the two equivalent.

PROPOSITION 2. *Let an inequality measure I satisfy WD. Then PP holds if and only if N holds for I.*

Proof. If I satisfies WD, then where $n \geqslant 1$, $z \in D_n$, $m \geqslant 2$, and y is a replication of z (m times), we have $I(y) = I(z) + I(\bar{z} u^m)$.

Propositions 1 and 2 suggest that the distinction between TD and WD is a real one. In fact there exist measures which satisfy WD and yet violate TD.[10] The next result shows that the combination of WD and PP is quite closely related to TD.

PROPOSITION 3. *Let an inequality measure I satisfy S. Then TD holds if and only if WD and PP hold for I.*

Proof. In the Appendix.

Hence, for symmetric measures, Theil decomposability is equivalent to weak Theil decomposability and the population principle.

[9] It can be seen that TD places constraints on the cardinal form of the inequality measure, while PD, S, H, and PP concern only the underlying inequality ordering. See [2, 15, 24] for discussions of measurability assumptions on inequality measures.

[10] The Shannon entropy measure (see footnote 12) applied to the vector of income shares satisfies WD, but not TD or PP. Instead of defining WD we could have defined a property, say, WD', which is in all respects the same as WD except that the between-group term is as in TD. Note, though, that unlike WD, the property WD' would *not* be independent of PP. In fact, the proof of Proposition 3 shows that WD' is *equivalent* to TD in the presence of symmetry.

5. Uniqueness Theorems

In the previous sections we have investigated the general properties that a measure of inequality is expected to have, and a particular type of decomposability exhibited by the Theil measure that is appealing both in its simplicity and in the interpretation of its within- and between-group inequality terms. We now present the main results of the paper which characterize the Theil measure of income inequality. For clarity of exposition the proofs of the next three results, and the more general result that implies them all, are presented in Section 7, below.

THEOREM 1. *An inequality measure I satisfies PD, S, H, and TD if and only if I is a positive multiple of the Theil measure of inequality.*

We noted before that symmetry, homogeneity, and the Pigou–Dalton transfers principle are commonly regarded as the basic requirements for an inequality measure to satisfy. Theorem 1 says that the Theil measure is essentially the only measure of inequality which satisfies Theil decomposability in addition to the basic properties. An interesting variant of this result is immediate if PP is also taken to be a basic requirement.

THEOREM 2. *An inequality measure I satisfies PD, S, H, PP, and WD if and only if I is a positive multiple of the Theil measure of inequality.*

We saw by example that the properties PD, S, and H are not sufficient to ensure that the measure of inequality gives coherent comparisons of inequality across population sizes. To obtain the desired consistency, one may take PP as a fourth basic property. Theorem 2 then tells us that the Theil measure is essentially the only measure which satisfies these four properties and WD as well.

6. Generalization of a Theorem of Bourguignon

Recent papers by Bourguignon [5], Shorrocks [25], and Cowell and Kuga [7, 8] have examined various decomposability properties and their implications for the form of inequality measures. In particular, Bourguignon has offered an interesting characterization of the Theil measure in terms of condition TD, but instead of taking the Pigou–Dalton transfers principle as a basic assumption, he takes:

SECOND-ORDER DIFFERENTIABILITY. For each $n \geqslant 1$, I^n has continuous second-order partial derivatives.

He then shows that the Theil measure is essentially the only second-order differentiable measure of inequality that satisfies D, S, H, and TD.[11]

This result represents a significant advance in the axiomatic characterization of inequality measures. However, the arguments leading to it rely crucially on second-order differentiability. While this condition might be justified in other areas of economic theory, in the present context it is rather suspect. First, many of the standard measures of inequality, most prominantly the Gini coefficient, are *not* differentiable. Second, as a criterion for choosing among inequality measures, second-order differentiability carries remarkably little economic meaning: "The only justification for this requirement is that it is satisfied by all the usual measures except at some particular points" [5, p. 902].

As a result of the generality of the technique used to prove Theorems 1 and 2, we can rid Bourguignon's result of the objectionable differentiability assumption, and replace it with the following weaker continuity assumption:

CONTINUITY (C). For each $n \geqslant 1$, I is continuous on D_n.

THEOREM 3. *An inequality measure I satisfies nontriviality, S, D, C, H, and TD if and only if I is a positive multiple of the Theil measure.*

7. A GENERAL THEOREM AND PROOFS

The Theil measure of inequality is derived from the Shannon measure of entropy in information theory.[12] Since its introduction in 1948, numerous axiomatic characterizations of the Shannon measure have been proposed, including the well-known development of Khinchin [18]. A particularly general result in this vein is based on the following theorem by Lee [20].

THEOREM (Lee). *If h: $(0, 1) \to \mathbb{R}$ satisfies:*

  (i) *$h(t)$ is Lebesgue measurable on $(0, 1)$,*

  (ii) *$h(t) = h(1 - t)$ on $(0, 1)$,*

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[11] Bourguignon [5, Proposition 5]. Note that Bourguignon assumes N as well, which is not needed by Proposition 1. Additionally, he omits nontriviality and D; his results are not quite true unless assumptions such as these are included. Shorrocks [25] has independently established the more general result that second-order differentiability, D, S, H, and a general additive form of decomposition (where within-group weights may vary) together yield a general class of decomposable measures (including the Theil measure). For alternative characterizations of the Theil measure see Cowell and Kuga [7, 8].

[12] The Shannon measure H is defined by $H(x) = -\sum_{i=1}^{n} x_i \ln(x_i)$ (and $0 \ln(0) \equiv 0$), where $x = (x_1,...,x_n)$ is any probability vector. The Theil measure $T(y)$ is simply the difference between $\ln(n)$ and the Shannon measure applied to the vector $y/|y|$ of income shares.

(iii) $h(s) + (1 - s) h(t/(1 - s)) = h(t) + (1 - t) h(s/(1 - t))$ *for   all* $s, t > 0$ *such that* $s + t < 1$,

then $h(t)$ *is a multiple of* $t \ln(t) + (1 - t) \ln(1 - t)$.

We shall exploit this theorem and the relationship between the Theil and Shannon measures to prove the following general uniqueness theorem for the Theil measure.

THEOREM 4.   *Let* I *be a nontrivial inequality measure such that for* $f: [0, 1] \to \mathbb{R}$ *defined by* $f(t) = I(t, 1 - t)$:

   (i) $f(t)$ *is Lebesgue measurable on* $(0, \frac{1}{2})$,

   (ii) $f(t) = f(1 - t)$ *on* $[0, \frac{1}{2})$,

   (iii) $f(t) = I(\alpha t, \alpha(1 - t))$ *on* $[0, \frac{1}{2}]$ *for all* $\alpha > 0$,

   (iv) $f(0) \geqslant f(\frac{1}{2})$.

If I *satisfies TD, then* I *is a positive multiple of the Theil measure of inequality.*

Theorem 4 states that under rather minimal assumptions, Theil decomposability characterizes the Theil measure of inequality (up to a positive multiple). The proof of Theorem 4 uses the following two lemmas. (Their proofs are in the Appendix.)

LEMMA 1.   *Suppose an inequality measure* I *satisfies TD, and* $I(y_1, y_2) = I(y_2, y_1)$ *for all* $(y_1, y_2) \in D_2$. *Then for all* $n \geqslant 2$ *we have* $I(y) = I(yP)$ *for each* $y \gg 0^n$ *in* D_n *and every permutation matrix* P *of order* n.

LEMMA 2.   *Suppose an inequality measure* I *satisfies TD,* $I(y_1, y_2) = I(y_2, y_1)$ *for all* $(y_1, y_2) \in D_2$, *and* I *is continuous on* D_2. *Then* I *satisfies* C *and* S.

In words, Lemma 1 indicates that TD extends symmetry for two-person distributions to symmetry over all *positive* distributions of arbitrary population size. We do not necessarily obtain symmetry over *all* distributions, though, because TD does not apply to partitions where some $y^{(i)}$ is a vector of zeros. Lemma 2 shows that TD transforms the two-person analogs of continuity and symmetry into conditions C and S, which hold for all distributions in $\mathcal{D}$. We now prove Theorem 4.

Proof.   Let $I: \mathcal{D} \to R$ satisfy TD and (i)–(iv). Define $\hat{h}: [0, 1] \to \mathbb{R}$ by $\hat{h}(t) = f(0) - f(t)$ and let $h: (0, 1) \to \mathbb{R}$ be its restriction to the open interval $(0, 1)$. We shall show that $h(t)$ satisfies the hypothesis of Lee's Theorem.

Assumptions (i) and (ii) on $f(t)$ immediately imply that $h(t)$ satisfies (i) and (ii) of Lee's Theorem; only (iii) remains to be verified. Let $s, t > 0$ be such that $r \equiv 1 - s - t > 0$, and consider the distributions $v = (s, 0, r, t)$ and $w = (t, 0, r, s)$. TD implies that $I(v) = I(w)$, since $I(s, r/2, r/2, t) = I(t, r/2, r/2, s)$ by Lemma 1. Furthermore, by an additional application of TD,

$$I(v) = sI(s, 0) + (1 - s)I(r, t) + I\left(\frac{s}{2}, \frac{s}{2}, \frac{1 - s}{2}, \frac{1 - s}{2}\right). \tag{4}$$

Now $I(s, 0) = f(0)$ and $I(r, t) = f(r/(1 - s))$ by assumptions (ii) and (iii), while $I(s/2, s/2, (1 - s)/2, (1 - s)/2) = f(s)$ by Lemma 1, Proposition 1, and (iii). Thus $I(v) = sf(0) + (1 - s)f(t/(1 - s)) + f(s)$. By definition of $h(t)$, then, $h(s) + (1 - s)h(t/(1 - s)) = 2f(0) - I(v)$. Similar arguments lead to $I(w) = tf(0) + (1 - t)f(s/(1 - t)) + f(t)$ and $h(t) + (1 - t)h(s/(1 - t)) = 2f(0) - I(w)$. Therefore, $h(t)$ satisfies all requirements of Lee's Theorem, and it follows that $h(t)$ is a multiple of $t \ln(t) + (1 - t)\ln(1 - t)$, say $h(t) = -\lambda[t \ln(t) + (1 - t)\ln(1 - t)]$ for some $\lambda \in \mathbb{R}$.

Clearly $\hat{h}(0) = 0$, and so $\hat{h}(1) = 0$ as well by assumption (ii). Thus, $\hat{h}(t) = -\lambda[t \ln(t) + (1 - t)\ln(1 - t)]$ for all $t \in [0, 1]$. Note that $\hat{h}(\frac{1}{2}) = \lambda \ln(2)$. Since Proposition 3 implies $f(\frac{1}{2}) = 0$, we know that $f(0) = f(1) = \lambda \ln 2$ by the definition of $\hat{h}(t)$. Therefore, $f(t) = \lambda \ln(2) + \lambda[t \ln(t) + (1 - t)\ln(1 - t)] = \lambda[t \ln(2t) + (1 - t)\ln(2(1 - t))]$. Hence, by assumption (iii),

$$I(x) = \lambda \sum_{i=1}^{2} \frac{x_i}{|x|} \ln\left(2 \frac{x_i}{|x|}\right), \qquad x \in D_2, \tag{5}$$

for some $\lambda \in \mathbb{R}$. An immediate implication of this result and Lemma 2 is that I satisfies C and S.

Now suppose that for a given $m \geqslant 1$ we have

$$I(x) = \lambda \sum_{i=1}^{2^k} \frac{x_i}{|x|} \ln\left(2^k \frac{x_i}{|x|}\right), \qquad x \in D_{(2^k)}, \tag{6}$$

for every $k = 1, \ldots, m$. We shall show that this holds for $k = m + 1$ as well. Denote $n = 2^m$ and $n' = 2^{m+1}$. Let $y \in D_{n'}$ have only strictly positive entries, and let $x, z \in D_n$ be such that $y = (x; z)$. By Proposition 3, $I(y) = (|x|/|y|)I(x) + (|z|/|y|)I(z) + I(\bar{x}, \bar{z})$, so that by the induction hypothesis

$$I(y) = \lambda \frac{|x|}{|y|} \sum_{i=1}^{n} \frac{x_i}{|x|} \ln\left(n \frac{x_i}{|x|}\right) + \lambda \frac{|z|}{|y|} \sum_{i=1}^{n} \frac{z_i}{|z|} \ln\left(n \frac{z_i}{|z|}\right)$$

$$+ \lambda \frac{|x|}{|y|} \ln\left(2 \frac{|x|}{|y|}\right) + \lambda \frac{|z|}{|y|} \ln\left(2 \frac{|z|}{|y|}\right), \tag{7}$$

where we have used the identities $\bar{x}/(\bar{x} + \bar{z}) = |x|/|y|$ and $\bar{z}/(\bar{x} + \bar{z}) = |z|/|y|$. However, since $\ln(nx_i/|x|) = \ln(n'x_i/|y|) - \ln(2|x|/|y|)$ and $\ln(nz_i/|z|) = \ln(n'z_i/|y|) - \ln(2|z|/|y|)$, we have

$$I(y) = \lambda \sum_{i=1}^{n} \frac{x_i}{|y|} \ln \left(n' \frac{x_i}{|y|}\right) + \lambda \sum_{i=1}^{n} \frac{z_i}{|y|} \ln \left(n' \frac{z_i}{|y|}\right)$$

$$= \lambda \sum_{i=1}^{n'} \frac{y_i}{|y|} \ln \left(n' \frac{y_i}{|y|}\right). \tag{8}$$

as desired. By C, this result holds for arbitrary $y \in D_{n'}$, and hence we have shown that

$$I(x) = \lambda \sum_{i=1}^{2^k} \frac{x_i}{|x|} \ln \left(2^k \frac{x_i}{|x|}\right), \qquad x \in D_{(2^k)}, \tag{9}$$

for every positive integer k.

Now to complete the proof, let x be an arbitrary element of D_n for $n \geqslant 1$. Clearly, there exists some integer $l \geqslant 1$ such that $n + l = m$, where $m = 2^k$ for some $k \geqslant 2$. Define $y = (x; \bar{x}u^l)$ and note that $I(y) = (n/m) I(x)$ by TD and Proposition 1. Since $y \in D_m$ we know that

$$I(y) = \lambda \sum_{i=1}^{m} \frac{y_i}{|y|} \ln \left(m \frac{y_i}{|y|}\right) \tag{10}$$

and so

$$I(y) = \lambda \sum_{i=1}^{n} \frac{x_i}{|y|} \ln \left(m \frac{x_i}{|y|}\right) + \lambda \sum_{i=1}^{l} \frac{\bar{x}}{|y|} \ln \left(m \frac{\bar{x}}{|y|}\right). \tag{11}$$

The identity $|y| = m\bar{x} = m|x|/n$ yields

$$I(y) = \frac{n}{m} \lambda \sum_{i=1}^{n} \frac{x_i}{|x|} \ln \left(n \frac{x_i}{|x|}\right). \tag{12}$$

Hence, I is of the form

$$I(x) = \lambda \sum_{i=1}^{n} \frac{x_i}{|x|} \ln \left(n \frac{x_i}{|x|}\right), \qquad x \in D_n, \tag{13}$$

for some $\lambda \in \mathbb{R}$. The nontriviality of I and assumption (iv) imply that $\lambda > 0$, which concludes the proof.

Remark. There is no attempt made here to justify the assumptions of Theorem 4 in economic terms. For instance, Lebesgue measurability of $f(t)$, per se, may be of little direct economic interest; but the fact that the class of

such functions is so large as to include, say, all monotonic functions and all continuous functions, will lead to results that are clearly economically meaningful. The above theorem is to be seen as a technical result of interest mainly for its generality. We shall now use it to prove Theorems 1 and 3. Theorem 2 follows from Proposition 3 and Theorem 1.

Proof of Theorem 1. Suppose $I: \mathscr{D} \to \mathbb{R}$ satisfies PD, S, H, and TD. As in Theorem 1 define $f: [0, 1] \to \mathbb{R}$ by $f(t) = I(t, 1 - t)$ on $[0, 1]$. Property S implies that $f(t) = f(1 - t)$ on $[0, 1]$, while property H implies $f(t) = I(t, 1 - t) = I(\alpha t, \alpha(1 - t))$ for any $t \in [0, 1]$ and $\alpha > 0$. Hence, requirements (ii) and (iii) of Theorem 4 are satisfied, leaving nontriviality, (i), and (iv) to be verified.

To this end, note that $f(t)$ is strictly decreasing on $[0, \frac{1}{2}]$ and strictly increasing on $[\frac{1}{2}, 1]$ by PD. Nontriviality and (iv) follow directly from this observation. Additionally, (i) is obtained, since for each $\beta \in \mathbb{R}$ the set $\{t \in (0, 1): f(t) < \beta\}$ is either empty, a single point, or an interval, all Lebesgue measurable sets. Thus, I is the Theil measure by Theorem 4. The converse is immediate.

Proof of Theorem 3. Let $I: \mathscr{D} \to \mathbb{R}$ be nontrivial and satisfy S, D, C, H, and TD. Then (i), (ii), (iii), and (iv) of Theorem 4 are implied by C, S, H, and D, respectively, and so I is the Theil measure. The converse is immediate.

Remark. It is important to note that Theorems 1 and 2 are logically independent of Theorem 3: Since conditions PD, S, and H do not imply continuity, it does not follow from Theorem 3 that T is the only measure satisfying PD, S, H, and TD. Similarly, Theorem 1 does not imply Theorem 3. Furthermore, any attempt to extend Bourguignon's result directly to Theorem 3 by approximation methods is not likely to be successful. In order to show the nonexistence of an alternative to the Theil measure among all measures satisfying the assumptions of Theorem 3, one would typically assume that such a function exists, construct a sequence of (second-order) differentiable functions converging to it, and then try to arrive at a contradiction using Bourguignon's theorem. The problem is that, while this theorem would tell us that each approximating function cannot satisfy the six properties, we are unable to draw the same conclusion for the limit function. The negation of the TD property is not preserved by the limiting process: a sequence of functions that violate TD may very well converge to a function which satisfies TD.

8. CONCLUDING REMARKS

The results of Atkinson [2], Dasgupta *et al.* [11], Rothschild and Stiglitz [23], and Fields and Fei [14] allow interesting interpretations of Theorems 1 and 2 in terms of the familar Lorenz criterion.[13] To every distribution x there corresponds a Lorenz curve, depicting the percentage of income received by the poorest t percent ($0 \leqslant t \leqslant 100$) of the population in x. (See Gastwirth [16] for a rigorous definition.) A distribution x is said to Lorenz (weakly) dominate a distribution y, written xLy, if the Lorenz curve of x lies nowhere below that of y.

It seems quite natural to require measures of inequality to be consistent with the Lorenz criterion when it applies; i.e., for every $x, y \in \mathscr{D}$ we have (i) xLy and yLx imply $I(x) = I(y)$ and (ii) xLy and not yLx imply $I(x) < I(y)$. A still weaker property is to demand consistency within each population size; i.e., for each n and every $x, y \in D_n$ we have (i) and (ii). It follows from the results of Atkinson *et al.*, that the class of Lorenz consistent measures comprises exactly those measures satisfying the Pigou–Dalton condition, symmetry, homogeneity, and the population principle, while the first three of these identify all measures consistent with the Lorenz criterion at each population size. (See [15] for details.)

Given these results, our first theorem says that "Lorenz consistency at each population size" and "Theil decomposability" completely characterize the Theil measure of income inequality. Our second theorem shows how the consistency requirement may be strengthened and the decomposability property weakened to obtain another characterization: "weak Theil decomposability" uniquely determines the Theil measure from among all "Lorenz consistent" measures of inequality.

APPENDIX

Proof of Proposition 3. Let $I: \mathscr{D} \to \mathbb{R}$ satisfy S. Suppose first that I exhibits property TD. Then I satisfies PP by Proposition 1. Let $y = (y^{(1)}; ...; y^{(m)})$, where $m \geqslant 2$, $n \geqslant 1$, and $y^{(i)} \in D_n$ for $i = 1,..., m$. Clearly, $I(\bar{y}^{(1)} u^n; ...; \bar{y}^{(m)} u^n) = I(\bar{y}^{(1)},..., \bar{y}^{(m)})$ by S and PP, hence I satisfies WD as well.

Now suppose that I satisfies WD and PP. Consider any $m \geqslant 2$ and $n_1,..., n_m \geqslant 1$. Let $y = (y^{(1)}; ...; y^{(m)})$, where $y^{(i)} \in D_{n_i}$ for $i = 1,..., m$, and denote $l = n_1 \cdot n_2 \cdots n_m$ and $l_i = l/n_i$. Let $x^{(i)}$ be an l-replication of $y^{(i)}$ for $i = 1,..., m$, and let $x = (x^{(1)}; ...; x^{(m)})$; further, let $z^{(i)}$ be an l_i-replication of

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[13] A particularly clear exposition of the key result may be found in Marshall and Olkin [21].

the n_i-distribution $y^{(i)}$. Then $z^{(i)} \in D_l$ and $x^{(i)}$ is an n_i-replication of $z^{(i)}$ for each i. We may apply WD to x partitioned into the subdistributions $z^{(i)}$:

$$I(x) = \sum_{i=1}^{m} \left[n_i \frac{|z^{(i)}|}{|x|} I(z^{(i)}) \right] + I(\bar{z}^{(1)} u^{n_1}; \ldots; \bar{z}^{(m)} u^{n_m}). \qquad (A.1)$$

Note that $I(z^{(i)}) = I(y^{(i)})$ and $I(x) = I(y)$ by S and PP, while $|z^{(i)}|/|x| = |y^{(i)}|/(n_i |y|)$, and $\bar{z}^{(i)} = \bar{y}^{(i)}$ for each i. Thus,

$$I(y) = \sum_{i=1}^{m} \frac{|y^{(i)}|}{|y|} I(y^{(i)}) + I(\bar{y}^{(1)} u^{n_1}; \ldots; \bar{y}^{(m)} u^{n_m}), \qquad (A.2)$$

from which TD follows.

Proof of Lemma 1. Let $I: \mathscr{D} \to \mathbb{R}$ be such that $I(y_1, y_2) = I(y_2, y_1)$ for all $(y_1, y_2) \in D_2$. The proof consists of showing that inequality is the same for any two strictly positive n-distributions that can be derived from one another by a transposition of adjacent entries. Once this has been verified, the result follows immediately from the fact that every permutation may be broken down into a sequence of such transpositions.[14]

Let $n \geqslant 3$, and consider any $y \gg 0^n$ in D_n. Choose any $k = 1, 2, \ldots, n - 1$, and define $x \in D_n$ by $x_k = y_{k+1}$, $x_{k+1} = y_k$, and $x_j = y_j$ for all $j \neq k, k + 1$. Then where $\hat{y} = (y_k, y_{k+1})$ and $\hat{x} = (x_k, x_{k+1})$, we have $I(y) - I(x) = (|\hat{y}|/|y|)[I(\hat{y}) - I(\hat{x})]$ by TD. Since $I(\hat{y}) = I(\hat{x})$, we obtain $I(y) = I(x)$.

Proof of Lemma 2. Let $I: \mathscr{D} \to \mathbb{R}$ satisfy the conditions specified in Lemma 2. It is clear that I is continuous on D_1. Now let $m \geqslant 3$, and suppose that I is continuous on D_j for $j = 1, 2, \ldots, m - 1$. We are to show that I is continuous on D_m as well.

Let $z^{(0)} \in D_m$ and let $\{z^{(l)}\}$ be a sequence in D_m converging to $z^{(0)}$. For $l = 0, 1, 2, \ldots$, define:

$$y^{(l)} = (z^{(l)}; \bar{z}^{(l)}) \qquad \text{if } m \text{ is odd and } z_k^{(0)} > 0 \text{ for some } k \leqslant (m + 1)/2;$$

$$= (\bar{z}^{(l)}; z^{(l)}) \qquad \text{if } m \text{ is odd and } z_k^{(0)} = 0 \text{ for all } k \leqslant (m + 1)/2;$$

$$= (\bar{z}^{(l)}; z^{(l)}; \bar{z}^{(l)}) \qquad \text{if } m \text{ is even.} \qquad (A.3)$$

Let m' be the number of entries in $y^{(l)}$. Note that by construction m' is even and $q \equiv m'/2 \leqslant m - 1$.

Let $w^{(l)}$ and $x^{(l)}$ be the vectors of length q such that $y^{(l)} = (w^{(l)}; x^{(l)})$. Surely $w^{(0)}$ and $x^{(0)}$ are nonnull, and since $w^{(l)} \to w^{(0)}$ and $x^{(l)} \to x^{(0)}$ it

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[14] For instance, see Herstein [17, p. 67]. Strictly speaking, his result concerns transpositions of coordinates that are not necessarily adjacent. However, it is a trivial matter indeed to see that any transposition may be broken down into a sequence of "adjacent coordinate" transpositions.

follows that for l large enough, $w^{(l)}$ and $x^{(l)}$ are nonnull as well. Thus, for $l = 0$ or l large enough we have $w^{(l)}, x^{(l)} \in D_q$ so that we may apply TD to obtain:

$$I(y^{(l)}) = \frac{|w^{(l)}|}{|y^{(l)}|} I(w^{(l)}) + \frac{|x^{(l)}|}{|y^{(l)}|} I(x^{(l)}) + I(\bar{w}^{(l)} u^q; \bar{x}^{(l)} u^q). \qquad \text{(A.4)}$$

By Proposition 1 and Lemma 1, $I(\bar{w}^{(l)}, \bar{x}^{(l)}) = I(\bar{w}^{(l)} u^q; \bar{x}^{(l)} u^q)$ for $l = 0$ or large l. Therefore, by the induction hypothesis $I(y^{(l)}) \to I(y^{(0)})$.

An additional application of TD and Proposition 1 yields $I(y^{(l)}) = (|z^{(l)}|/|y^{(l)}|) I(z^{(l)})$ for $l = 0, 1, 2,...$, where $|z^{(l)}|/|y^{(l)}| = m/m' > 0$. Thus, $I(z^{(l)}) \to I(z^{(0)})$ and so I is continuous on D_m.

To see that S holds, let $n \geqslant 1$ and note that by Lemma 1, $I(y) = I(yP)$ for all $y \gg 0^n$ in D_n and every permutation matrix P of order n. Clearly, C extends this results to all distributions y in D_n, and hence I satisfies S.

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