Oxford Poverty & Human Development Initiative (OPHI)
Oxford Department of International Development
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OPHI WORKING PAPER NO. 51

On the Generalization and Decomposition of the Bonferroni Index[1]

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June 2012

Abstract

A simple algorithm is proposed which defines the Bonferroni index as the product of a row vector of individual population shares, a linear mathematical operator called the Bonferroni matrix and a column vector of income shares. This algorithm greatly simplifies the decomposition of the Bonferroni index by income sources or classes and population subgroups. The proposed algorithm also links the Bonferroni index to the concepts of relative deprivation and social welfare and leads to a generalization where the traditional Bonferroni and Gini indices are special cases. The paper ends with an empirical illustration based on EU-SILC data for the year 2008.

Keywords: Gini, inequality decomposition, population subgroups, relative deprivation, social welfare

JEL classification: D31, D63

1 A preliminary version of this paper was presented by Jacques Silber at the forty eighth meeting of the Italian Society of Economics, Demography and Statistics, which took place in Rome on May 26-28 2011.

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This study has been prepared within the OPHI theme on multidimensionak poverty measurement.

OPHI gratefully acknowledges support from the UK Economic and Social Research Council (ESRC)/(DFID) Joint Scheme, Robertson Foundation, UNICEF N'Djamena Chad Country Office, Praus, Georg-August-Universität Göttingen, International Food Policy Research Institute (IFPRI), John Fell Oxford University Press (OUP) Research Fund, German Federal Ministry for Economic Cooperation and Development, United Nations Development Programme (UNDP) Human Development Report Office, national UNDP and UNICEF offices, and private benefactors. International Development Research Council (IDRC) of Canada, Canadian International Development Agency (CIDA), UK Department of International Development (DFID), and AusAID are also recognised for their past support.

ISSN 2040-8188 ISBN 978-1-907194-37-5

Acknowledgements

Jacques Silber completed this paper while he was visiting the Department of Economic Sciences of the University of Geneva, Switzerland, which he thanks for its very warm hospitality. He also gratefully acknowledges the financial support of the Adar Foundation of the Department of Economics of Bar-Ilan University.

Acronyms

EU-SILC: European Union Statistics on Income and Living Conditions

OECD: Organisation for Economic Co-operation and Development

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Many people consider the reduction of inequalities as a basic aim of society. Such ideas are, however, largely nonoperational, sterile, and even meaningless, as long as what is called inequality is not stated with precision….It thus seems essential to appraise the economic and, if we dare say, ethical, implications of the inequality measures, and to build measures embodying the economic and ethical properties we feel inequality means.

−Kolm (1976)

1. Introduction

Despite the great relevance of Kolm's (1976) remarks in the citation above, it is clear that the degree of popularity of the different inequality indices that have been proposed in the literature is related to the ease which with they can be grasped by a non-specialist. The Gini index thus remains the most commonly used index because it has a simple graphical interpretation in terms of the Lorenz curve, a diagrammatic tool that any layman can understand. This is also why it remains more popular than entropy related indices such as the two Theil indices or than the family of so-called Atkinson indices, not to mention the Bonferroni index whose definition even specialists of the field tend to ignore.

One may however wonder why the Bonferroni index has been overlooked for so many years. It is only in recent years that some authors attempted to rehabilitate this measure of inequality (e.g. Tarsitano, 1990; Aaberge, 2000; Giorgi and Crescenzi, 2001; Piesch, 2005; Aaberge, 2007; Chakravarty, 2007; Bárcena and Imedio, 2008; Imedio-Olmedo et al., 2011). Aaberge (2007) in particular drew our attention to several attractive properties of the Bonferroni index, or more precisely of what he called "scaled conditional mean curve" which is just another name for the Bonferroni (1930) curve, from which the definition of the Bonferroni index is derived. He thus stressed that since the scaled conditional mean for a given percentile p is the ratio of the mean income of the poorest $100p\%$ of the population and the overall mean, it yields important information on poverty, assuming the poverty line has been determined. Aaberge (2007) also emphasized the fact that in the case of a uniform distribution, the scale conditional mean curve becomes the diagonal and represents hence a useful reference line (in addition to the perfect equality line or to the case of maximal inequality). Aaberge showed in particular that if the Bonferroni curve intersects the diagonal once from below (single intersection), the corresponding distribution exhibits lower inequality than the uniform distribution below the intersection point and higher inequality than the uniform distribution above this intersection point. The Bonferroni curve which, like the Lorenz curve, is bounded by the unit square is also strongly related to the shape of the underlying distribution curve F: when F is convex (strongly skewed to the left), the Bonferroni curve is concave, and when F is concave (strongly skewed to the right), the Bonferroni curve is convex. Aaberge (2007) proved also that the Bonferroni index satisfies the principle of diminishing transfers (see Kolm, 1976, and Shorrocks and Foster, 1987) for all strictly log-concave distributional functions and the principle of positional transfer sensitivity (see Mehran, 1976) for all distributional functions. Finally the Bonferroni index may also be interpreted in terms of relative deprivation (see Chakravarty, 2007).

The Bonferroni curve and index seem therefore to have quite nice properties and one may indeed wonder why it did not become more popular. One reason is certainly related to the fact that the Bonferroni index does not obey Dalton's principle of population (see Bárcena and Imedio, 2008). A second element that seems to have been detrimental to the Bonferroni index is that its decomposition by income sources or population subgroups is rather cumbersome (see Tarsitano, 1990). The present paper will however propose a new algorithm to compute the Bonferroni index which should make such a breakdown much easier to implement as well as much more attractive. This paper is therefore an extension to the Bonferroni index of the results derived by Silber (1989) for the Gini index.

The following section thus shows that, as in the case of the Gini index (see Silber, 1989), the Bonferroni index can be expressed in matricial form via what we call the Bonferroni B-matrix. Section 2 then shows how the use of such a B-matrix greatly simplifies the decomposition of the Bonferroni inequality index by income sources. Sections 3 and 4, again using the B-matrix, then extend the analysis to the breakdown of inequality by income classes and by population subgroups. Section 5 presents some results on the link between the Bonferroni index and the concept of deprivation while section 6 gives a matricial expression to the social welfare function that lies behind the Bonferroni index. Section 7 proposes a generalization of the B-matrix and proves that both the Gini and the Bonferroni indices are special cases of such a generalization. Section 8 provides a short empirical illustration based on the European Union Statistics on Income and Living Conditions (EU-SILC) data set for the year 2008, while concluding comments are given in section 9.

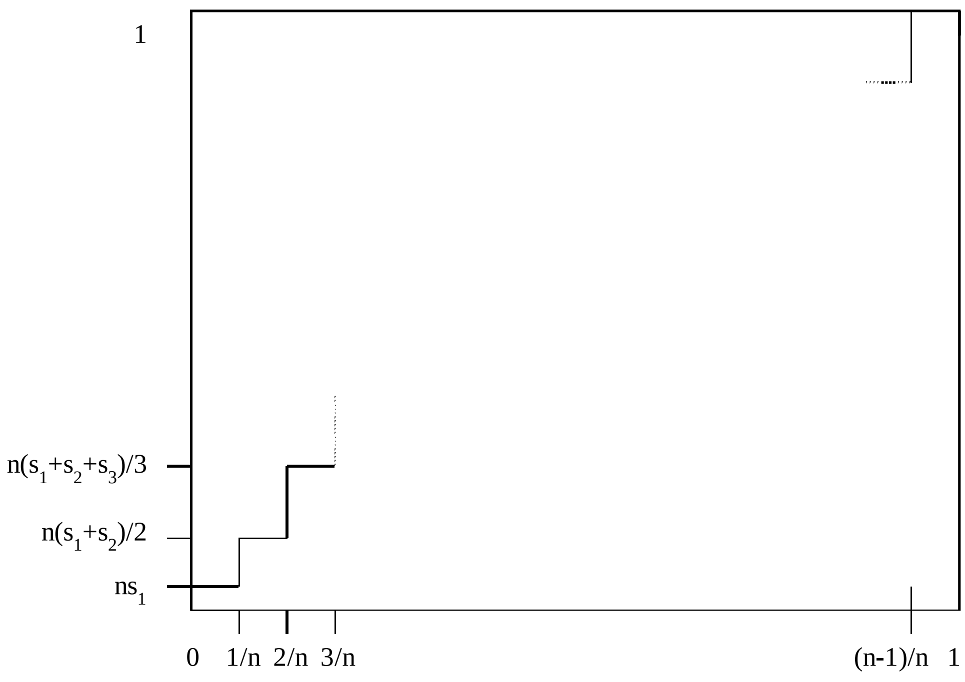
2. Defining a "B-matrix or "Bonferroni matrix"

The Bonferroni curve (see Bonferroni, 1930) is defined as follows. Assume n individuals, let x_i be the income of individual i arranged in ascending order. Let $s_i = (x_i / n\bar{x})$ be the share of income of individual i in total income, where $\bar{x}$ is the mean income in the whole population. Let us plot on the horizontal axis, like for the Lorenz curve, the cumulative population shares $\{(1/n),(2/n),...,(i/n),...((n-1)/n),1\}$. On the vertical axis we plot not the cumulative income shares (as in the case of the Lorenz curve) but the ratio of the cumulative income shares over the cumulative population shares. In other words we plot the following values:

$$\{(\frac{s_1}{(1/n)}),(\frac{(s_1+s_2)}{(2/n)}),...,(\frac{(s_1+s_2+...+s_i)}{(i/n)}),...,(\frac{(s_1+...+s_i+...+s_n)}{(n/n)}=\frac{1}{1}=1)\}$$

The Bonferroni index is then defined as the area lying between the Bonferroni curve and the horizontal line at height 1 (see Figure 1).

Figure 1: The Bonferroni curve



The Bonferroni index I_B is hence defined as

$$I_B = [(1/n)(1-(\frac{s_1}{(1/n)}))] + [(1/n)(1-(\frac{(s_1+s_2)}{(2/n)}))] + \dots [(1/n)(1-(\frac{(s_1+\dots+s_i)}{(i/n)}))] + \dots$$
$$+ [(1/n)(1-(\frac{s_1+\dots+s_{n-1})}{((n-1)/n)}))] + [(1/n)(1-(\frac{s_1+\dots+s_{n-1}+s_n)}{(n/n)}))] \qquad (1)$$

Let $\overline{x}_i = (\sum_{j=1}^{i} x_j)/i$ be the mean of the first i incomes which, as before, are ranked by increasing values. We may then rewrite (1) as

$$I_B = [(1/n)(1-(\frac{x_1}{\overline{x}}))] + [(1/n)(1-(\frac{(x_1+x_2)/2}{\overline{x}}))] + \dots [(1/n)(1-(\frac{(x_1+\dots+x_i)/i}{\overline{x}}))] + \dots$$
$$+ [(1/n)(1-(\frac{(x_1+\dots+x_{n-1})/(n-1)}{\overline{x}}))] + [(1/n)(1-(\frac{(x_1+\dots+x_{n-1}+x_n)/n}{\overline{x}}))] \qquad (2)$$

$$\Leftrightarrow I_B = (1/n)[(1-(\overline{x}_1/\overline{x})) + (1-(\overline{x}_2/\overline{x})) + \dots + (1-(\overline{x}_i/\overline{x})) + \dots + (1-(\overline{x}_{n-1}/\overline{x})) + (1-(\overline{x}/\overline{x}))] = \qquad (3)$$
$$= (1/n)(1/\overline{x})[(\overline{x}+\overline{x}+\dots+\overline{x}+\dots+\overline{x}+\overline{x}) + (\overline{x}_1+\overline{x}_2+\dots+\overline{x}_i+\dots+\overline{x}_{n-1}+\overline{x})] =$$
$$= (\overline{x} - (1/n)\sum_{i=1}^{n}\overline{x}_i)/\overline{x}$$

It is then easy to verify that expression (3) may be also written as

$$I_B = (B_A/\overline{x}) \qquad (4)$$

where $B_A = (\overline{x} - (1/n)\sum_{i=1}^{n}\overline{x}_i)$ refers to the absolute Bonferroni index of inequality (see Chakravarty, 2007).

Note that, using (4), B_A may be also expressed as

$$B_A = (1/n)\{[x_1+x_2+\dots+x_i+\dots+x_n] - [x_1 + ((x_1+x_2)/2) + \dots$$
$$+ ((x_1+x_2+\dots+x_i)/i) + \dots + ((x_1+x_2+\dots+x_i+\dots+x_n)/n)]\}$$

$$\leftrightarrow B_A = (1/n)\{[((x_1-x_1)/1)] + [((x_2-x_1)/2) + ((x_2-x_2)/2)] + \dots$$
$$+ [((x_i-x_1)/i) + \dots + ((x_i-x_i)/i)] + \dots + [((x_n-x_1)/n) + \dots + ((x_n-x_n)/n)]$$

$$\leftrightarrow B_A = (1/n)[\sum_{i=1}^{n}\sum_{j=1}^{i}((x_i - x_j)/i)] \tag{5}$$

This formulation of the absolute Bonferroni index was in fact already suggested by Chakravarty (2007).

Combining (4) and (5) we derive that

$$I_B = \sum_{i=1}^{n}\sum_{j=1}^{i}((s_i - s_j)/i) \tag{6}$$

Expression (6) may also be written as

$$\begin{aligned} I_B = {} & (s_1 - s_1) \\ & + ((s_2/2) - (s_1/2)) + ((s_2/2) - (s_2/2)) \\ & + \ldots \\ & + ((s_i/i) - (s_1/i)) + ((s_i/i) - (s_2/i)) + \ldots + ((s_i/i) - (s_i/i)) \\ & + \ldots \\ & + ((s_{n-1}/(n-1)) - (s_1/(n-1))) + ((s_{n-1}/(n-1)) - (s_2/(n-1))) + \ldots \\ & + ((s_{n-1}/(n-1)) - (s_i/(n-1))) + \ldots + ((s_{n-1}/(n-1)) - (s_{n-1}/(n-1))) \\ & + ((s_n/n) - (s_1/n)) + ((s_n/n) - (s_2/n)) + \ldots + ((s_n/n) - (s_i/n)) \\ & + \ldots + ((s_n/n) - (s_{n-1}/n)) + ((s_n/n) - (s_n/n)) \end{aligned} \tag{7}$$

and, as a consequence,

$$
\begin{aligned}
I_B = {} & s_1[-1-(1/2)-(1/3)-\ldots-(1/i)-\ldots-(1/(n-1))-(1/n)]+ \\
& +s_2[\quad -(1/2)-(1/3)-\ldots-(1/i)-\ldots-(1/(n-1))-(1/n)]+ \\
& +\ldots \\
& +s_i[\qquad -(1/i)-\ldots-(1/(n-1))-(1/n)]+ \\
& +\ldots \\
& +s_{n-1}[\qquad -(1/(n-1))-(1/n)]+ \\
& +s_n[\qquad -(1/n)]+ \\
& +s_1[1]+ \\
& +s_2[(1/2)+(1/2)]+ \\
& +\ldots \\
& +s_i[(1/i)+(1/i)+\ldots+(1/i)]+ \\
& +\ldots \\
& +s_{n-1}[(1/(n-1))+(1/(n-1))+\ldots+(1/(n-1))+\ldots+(1/(n-1))+(1/(n-1))]+ \\
& +s_n[(1/n)+(1/n)+\ldots+(1/n)+\ldots+(1/n)+(1/n)]
\end{aligned} \tag{8}
$$

It is then easy to derive that (8) may be also expressed in matricial form as

$$I_B = u'\tilde{B}s \tag{9}$$

where u' is a 1 by n row vector of n elements all equal to 1, s a n by 1 column vector of the income shares $s_1, s_2, \ldots, s_i, \ldots, s_{n-1}, s_n$ ranked by increasing values, and $\tilde{B}$ a n by n matrix which may be written as

$$
\tilde{B} = \begin{bmatrix}
0 & 1/2 & 1/3 & \ldots & 1/i & \ldots & 1/(n-1) & 1/n \\
-1/2 & 0 & 1/3 & \ldots & 1/i & \ldots & 1/(n-1) & 1/n \\
-1/3 & & 0 & \ldots & 1/i & \ldots & 1/(n-1) & 1/n \\
\ldots. & & & \ldots & \ldots & \ldots & 1/(n-1) & 1/n \\
-1/i & -1/i & -1/i & \ldots & 0 & \ldots & 1/(n-1) & 1/n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
-1/(n-1) & -1/(n-1) & -1/(n-1) & \ldots & -1/(n-1) & \ldots & 0 & 1/n \\
-1/n & -1/n & -1/n & \ldots & -1/n & \ldots & -1/n & 0
\end{bmatrix} \tag{10}
$$

Expression (9) may also be written as

$$I_B = e'Bs \tag{11}$$

where e' is a 1 by n row vector of the n individual population shares which are evidently all equal to $(1/n)$ and B, henceforth called the B-matrix or Bonferroni matrix, is defined as

$$B = n\tilde{B} \tag{12}$$

In other words B is defined as

$$B = \begin{bmatrix} 0 & n/2 & n/3 & \dots & n/i & \dots & n/(n-1) & n/n \\ -n/2 & 0 & n/3 & \dots & n/i & \dots & n/(n-1) & n/n \\ -n/3 & & 0 & \dots & n/i & \dots & n/(n-1) & n/n \\ \dots. & & & \dots & \dots & \dots & n/(n-1) & n/n \\ -n/i & -n/i & -n/i & \dots & 0 & \dots & n/(n-1) & n/n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -n/(n-1) & -n/(n-1) & -n/(n-1) & \dots & -n/(n-1) & \dots & 0 & n/n \\ -n/n & -n/n & -n/n & \dots & -n/n & \dots & -n/n & 0 \end{bmatrix} \tag{13}$$

Note that the B-matrix may be defined as follows: assuming that i refers to the line and j to the column, its typical element b_{ij} is equal to 0 if $i = j$, to $-(n/i)$ if $\mathrm{j} \prec \mathrm{i}$ and to (n/j) if $j \succ i$.[2] This clearly implies that $b_{ij} = -b_{ji}$ for $i \neq j$.

3. The decomposition of the Bonferroni index by income sources

Let us now call x_{ij} the income that individual i receives from income source j and assume that there are m sources of income. Let s_{ij} be the share $(\mathrm{x}_{\mathrm{ij}} / \sum_{\mathrm{j=1}}^{\mathrm{m}} \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x}_{\mathrm{ij}})$ which represents the share of the income that individual i receives from source j in the total income (all sources combined) in the population and let $s_i = \sum_{j=1}^{m} s_{ij}$ represent the share of total income that individual *i* receives from all

[2] Silber (1989) had defined in quite a similar way the Gini index, the linear mathematical operator being in this case what he called the $G-$ matrix.

sources. Call now $s_{.j}$ the column vector giving the shares s_{ij} for all the n individuals, these shares s_{ij} being ranked by increasing values of the shares s_i. The product $(e'Bs)$ in (11) may therefore be rewritten as

$$I_B = (e'Bs) = \sum_{j=1}^{m} (e'Bs_{.j}) \tag{14}$$

Let us also define a vector $\sigma_{.j}$ whose typical elements are equal to the shares $(s_{ij} / \sum_{i=1}^{n} s_{ij})$ of the various individuals in the total income from source j, these shares $(s_{ij} / \sum_{i=1}^{n} s_{ij})$ being ranked by increasing values of the shares s_i of the individuals in total income.

Let us similarly define a vector $v_{.j}$ whose typical elements are also equal to the shares $(s_{ij} / \sum_{i=1}^{n} s_{ij})$ of the various individuals in the total income from source j but these shares $(s_{ij} / \sum_{i=1}^{n} s_{ij})$ are now ranked by increasing values.

Given that $\sum_{j=1}^{m}\sum_{i=1}^{n} s_{ij} = 1$, it is easy to derive that expression (11) may be also written as

$$I_B = (e'Bs) = \sum_{j=1}^{m} \{ \left(\frac{(\sum_{i=1}^{n} s_{ij})}{(\sum_{j=1}^{m}\sum_{i=1}^{n} s_{ij})} \right) (e'Bv_{.j}) \left(\frac{(e'B\sigma_{.j})}{(e'Bv_{.j})} \right) \} \tag{15}$$

The ratio $\left(\frac{(e'B\sigma_{.j})}{(e'Bv_{.j})} \right)$ could be labeled "Bonferroni correlation coefficient" since it compares the value of the Bonferroni index of the shares $(s_{ij} / \sum_{i=1}^{n} s_{ij})$ of the various individuals in the income from source j (this is the expression $e'Bv_{.j}$) with the value of what could be called a "Pseudo-Bonferroni index" which is obtained when we rank the shares $(s_{ij} / \sum_{i=1}^{n} s_{ij})$ not by increasing values but by increasing values of the shares s_i of the various individuals in total income[3].

Expression (15) shows therefore that the Bonferroni index can be expressed as the sum over all income sources j of the product of three elements:

- the share $\left(\frac{(\sum_{i=1}^{n} s_{ij})}{(\sum_{j=1}^{m} \sum_{i=1}^{n} s_{ij})} \right)$ of income source j in total income
- the Bonferroni index $(e' B v_{.j})$ for income source j
- the "Bonferroni correlation coefficient" $\left(\frac{(e' B \sigma_{.j})}{(e' B v_{.j})} \right)$ for income source j which measures in a way the correlation between income source j and total income[4].

4. The decomposition of the Bonferroni index by income classes

Let us now assume that the (total) income data are available by income classes. There are K income classes (e.g. 10 deciles) and in each income class h there are n_h individuals so that $n = \sum_{h=1}^{K} n_h$. Let us also partition the B-matrix into K^2 submatrices so that the B-matrix may be written as

[3] The expression "Pseudo-Bonferroni index" is evidently used to stress the parallelism with what is known as the "Pseudo-Gini index" (see, Silber, 1989).

[4] The result shown by expression (15) is evidently very similar to the one obtained when decomposing the Gini index by income source (see, Silber, 1989).

$$B = \begin{bmatrix} B(n_1, n_1) & \ldots & B(n_1, n_K) \\ . & & . \\ . & & . \\ . & & . \\ B(n_K, n_1) & \ldots & B(n_K, n_K) \end{bmatrix} \quad (16)$$

Note that the K square matrices $B(n_h, n_h)$ of size n_h by n_h have zeros on their diagonal. Whenever $j \succ i$, any cell (i, j) in these matrices will be equal to (n/j) while whenever $(\mathrm{j} \prec \mathrm{i})$ any cell (i, j) in these matrices will be equal to $-(\mathrm{n}/\mathrm{i})$.

For the $K(K-1)$ remaining matrices $B(n_p, n_q)$ of size p by q we can observe that the element in the cell (i, j) will be equal to (n/j) whenever $j \succ i$, i and j referring respectively to the rank of the line and the column in the original B-matrix of size n by n. Similarly in these $B(n_p, n_q)$ matrices the element in the cell (i, j) will be equal to $-(\mathrm{n}/\mathrm{i})$ whenever $j \prec i$, i and j referring here also to the rank of the line and the column in the original B-matrix of size n by n.

Let us now similarly decompose the row vector e' and the column vector s in (11) into K components respectively called $e'(n_h)$ and $s(n_h)$, having each n_h elements. The product $e'Bs$ defined in (11) may now be written, using well-known rules on partitioned matrices, as

$$e'Bs = \sum_{p=1}^{K} e'(n_p) B(n_p, n_p) s(n_p) + \sum_{p=1}^{K} [\sum_{q \neq p}^{K} e'(n_p) B(n_p, n_q) s(n_q)] \quad (17)$$

Using (17) it is then easy to derive that the within-income classes inequality B_{WITH} is expressed as

$$B_{WITH} = \sum_{p=1}^{K} e'(n_p) B(n_p, n_p) s(n_p) \quad (18)$$

while the between income classes inequality B_{BET} will be written as

$$B_{BET} = \sum_{p=1}^{K} [\sum_{q \neq p}^{K} e'(n_p) B(n_p, n_q) s(n_q)] \quad (19)$$

Let us now define as B_{n_p} the Bonferroni matrix one would obtain in the case of n_p individuals (observations). It should be clear that such a matrix B_{n_p} is generally different from the matrix $B(n_p, n_p)$ defined previously, the only exception being in the case where the n_p individuals are the n_p poorest individuals, that is, when the n_p by n_p matrix B_{n_p} is identical to the matrix composed of the n_p first lines and n_p first columns of the overall Bonferroni matrix of size n by n.We can however express the within-income classes B_{WITH} contribution to the overall Bonferroni index as

$$B_{WITH} = \sum_{p=1}^{K} [e'(n_p) B(n_p, n_p) s(n_p)] = \sum_{p=1}^{K} \{(n_p / n)(n_p \bar{s}_p)(e'_{n_p} B_{n_p} s_{n_p}) \frac{[e'_{n_p} B(n_p, n_p) s_{n_p}]}{(e'_{n_p} B_{n_p} s_{n_p})}\} \quad (20)$$

where $\bar{s}_p$ is the average individual income share in income class p, $\bar{s}_p = \frac{\bar{x}_p}{\bar{x} n}$, $\bar{x}_p$ is the mean income in the income class p, e'_{n_p} a row vector of n_p elements , each being equal to $(1/n_p)$ and s_{n_p} a column vector of n_p elements whose typical element s_{i,n_p} is equal to the share of individual i in the total income of the income class p. If we call v_p the population share (n_p / n) of income class p, w_p the income share $n_p \bar{s}_p$, B_p the Bonferroni index for group p and C_p the "correction factor" for group p defined as

$$C_p = \frac{[e'_{n_p} B(n_p, n_p) s_{n_p}]}{(e'_{n_p} B_{n_p} s_{n_p})} \quad (21)$$

we can rewrite (20) as

$$B_{WITH} = \sum_{p=1}^{K} [v_p w_p B_p C_p] \quad (22)$$

In other words the within-income classes Bonferroni index is equal to the sum over all K income classes of the product of the population share, income share, Bonferroni index and correction factor[5] for each income class p [6].

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[5] See the appendix for a detailed formulation of the numerator and denominator of this correction factor.

[6] One may remember that the within-income classes Gini index is equal to the sum over all income classes of the population share, income share and Gini index for each income class p (see, Silber, 1989). There is thus no "correction factor" in the case of the within-income classes Gini index.

It is important to note that B_{WITH} is not equal to the weighted sum of the Bonferroni indices of the different income classes since expression (22) may be also written as

$$B_{WITH} = \sum_{p=1}^{K} [v_p w_p B_p C_p] = \sum_{p=1}^{K} v_p w_p B_p + \sum_{p=1}^{K} v_p w_p B_p (C_p - 1) = B^w + B^{residual\ W} \qquad (23)$$

where B^w is equal to the sum of the within-income classes Bonferroni indices while the residual term $B^{residual\ W}$ is a consequence of the fact that the Bonferroni index does not obey Dalton's principle of population.

Let us now take a look at the definition of the between-income classes inequality B_{BET} defined in (19).

Let us in particular express the two components of this between-income classes inequality which deal with the comparison of groups p and q. Using (19) the sum S_{pq} of these two components may be written as

$$S_{pq} = [e'(n_p)B(n_p, n_q)s(n_q)] + [e'(n_q)B(n_q, n_p)s(n_p)] \qquad (24)$$

As an illustration, assume that there are five individuals. Income class p is the first (poorest) income class and it includes the first two individuals (the two poorest individuals) while income class q will be assumed to be the second income class and to include the next three individuals. It is then easy to verify that

$$[e'(n_p)B(n_p, n_q)s(n_q)] = \begin{bmatrix}(1/n) & (1/n)\end{bmatrix} \begin{bmatrix}(n/3) & (n/4) & (n/5) \\ (n/3) & (n/4) & (n/5)\end{bmatrix} \begin{bmatrix}s_3 \\ s_4 \\ s_5\end{bmatrix}$$

Similarly one can write that

$$[e'(n_q)B(n_q, n_p)s(n_p)] = \begin{bmatrix}(1/n) & (1/n) & (1/n)\end{bmatrix} \begin{bmatrix}-(n/3) & -(n/3) \\ -(n/4) & -(n/4) \\ -(n/5) & -(n/5)\end{bmatrix} \begin{bmatrix}s_1 \\ s_2\end{bmatrix}$$

It is then easy to derive that the sum $S_{pq} = [e'(n_p)B(n_p, n_q)s(n_q)] + [e'(n_q)B(n_q, n_p)s(n_p)]$

will be expressed as

$$\mathrm{S_{pq}} = \{(2/\mathrm{n})[(\mathrm{n}/3)\mathrm{s}_3 + (\mathrm{n}/4)\mathrm{s}_4 + (\mathrm{n}/5)\mathrm{s}_5] -$$
$$-\{(1/\mathrm{n})[(\mathrm{n}/3)(\mathrm{s}_1+\mathrm{s}_2) + (\mathrm{n}/4)(\mathrm{s}_1+\mathrm{s}_2) + (\mathrm{n}/5)(\mathrm{s}_1+\mathrm{s}_2)]\}$$

$$\Leftrightarrow S_{pq} = (1/3)(s_3 - s_1) + (1/3)(s_3 - s_2) + (1/4)(s_4 - s_1) +$$
$$+(1/4)(s_4 - s_2) + (1/5)(s_5 - s_1) + (1/5)(s_5 - s_2)\}$$

The latter expression clearly shows that the sum S_{pq} amounts to an across groups p and q comparison since each of the income shares s_3, s_4 and s_5 of group q is compared with each of the income shares s_1 and s_2 of group p.

Using notations introduced previously, let us call $\bar{s}_p$ $(\bar{s}_q)$ the average income share in income class p (q), e'_{n_p} (e'_{n_q}) a row vector of n_p (n_q) elements , being each equal to $1/\mathrm{n_p}$ $(1/\mathrm{n_q})$ and finally s_{n_p} (s_{n_q}) a column vector with n_p (n_q) elements whose typical elements s_{i,n_p} (s_{i,n_q}) is equal to the share of individual i in the total income of income class p (q). If we call v_p and v_q the population shares $\mathrm{n_p}/\mathrm{n}$ and $\mathrm{n_q}/\mathrm{n}$ of income classes p and q and w_p and w_q the corresponding income shares $\mathrm{n_p}\bar{\mathrm{s}}_\mathrm{p}$ and $\mathrm{n_q}\bar{\mathrm{s}}_\mathrm{q}$, it is easy to derive, combining (19) and (24) that the between-income classes inequality index B_{BET} may be expressed as

$$B_{BET} = \sum_{p=1}^{m}\sum_{q\succ p}^{m}\{(v_p w_q)[e'_{n_p} B(n_p, n_q)s_{n_q}] + (v_q w_p)[e'_{n_q} B(n_q, n_p)s_{n_p}]\} \tag{25}$$

If we now call B_{pq} the expression $[e'_{n_p} B(n_p, n_q)s_{n_q}]$ and B_{qp} the expression $[e'_{n_q} B(n_q, n_p)s_{n_p}]$, we may conclude, on the basis of (25), that the between-income classes Bonferroni index B_{BET} is equal to the sum of the weighted average of the between groups p and q Bonferroni indices, B_{pq} and B_{qp}, the weights being equal to the product of the population and income shares of groups p and q.

5. The decomposition of the Bonferroni index by population subgroups: the case of overlapping subgroups

Let us now assume that there are m population subgroups whose incomes may overlap, in the sense that if, for example, group 1 is the poorest group (the group whose average income is the lowest of all groups) and group 2 the second poorest group, some individuals belonging to group 1 may earn more than some individuals belonging to group 2 while some individuals belonging to group 2 may earn less than some individuals belonging to group 1. Let the row vector e' refer, as before, to the population shares $(1/n)$ of each individual and let s represent the vector of individual income shares s_i ranked by increasing values of these incomes.

The Bonferroni index for the whole population, say, $I_{B,\,TOTAL}$, will therefore be defined, as before, as

$$I_{B,\,TOTAL} = e'\,Bs \qquad (26)$$

Let us now define a vector τ whose shares τ_i are ranked first by increasing values of the average income of the population subgroup to which the individuals belong, and within each population subgroup by increasing individual incomes. Assume that there are m population subgroups and n_h individuals in each population subgroup h. Call x_{hj} the income of individual j who belongs to population subgroup h and τ_{hj} the share of his/her income in the total income of the society. Finally call $\bar{x}_h = ((\sum_{j=1}^{n_h} x_{hj})/n_h)$ the average income of the individuals belonging to population subgroup h and $\overline{s_h} = (\bar{x}_h / \bar{x}n)$ the average income share (in the total income of society) of the individuals belonging to group h where $\bar{x}$ refers to the average individual income in society as a whole and $n = \sum_{h=1}^{m} n_h$ to the total number of individuals in the population. Let $\bar{\tau}$ be the vector of the income shares $\overline{s_h}$ where the first n_1 elements are all equal to $\overline{s_1}$, the next n_2 elements to $\overline{s_2}$, etc…

We therefore assume that

$$\tau_{h1} \le \ldots \le \tau_{hi} \le \ldots \le \tau_{hn_h} \quad \forall h$$

and that

$$\bar{\tau}_1 \le \ldots \le \bar{\tau}_h \le \ldots \le \bar{\tau}_K .$$

Using the results of section 4 we may easily conclude that the product $e'B\tau$ is actually equal to the sum of the within and between-groups inequality. The overall Bonferroni index was however defined in (26) as being equal to $e'Bs.$ Given that the Bonferroni matrix is by definition a linear operator, we may write that

$$\begin{aligned} I_{B,\,TOTAL} &= e'Bs = (e'B\tau) + [(e'Bs) - (e'B\tau)] = [(e'B\tau) - (e'B\bar{\tau})] + (e'B\bar{\tau}) + [(e'Bs) - (e'B\tau)] = \\ &= B_{WITH} + B_{BET} + OV \end{aligned} \qquad (27)$$

where B_{WITH}, B_{BET} and OV measure respectively the within groups inequality, the between-groups inequality and the degree of overlap between the income distributions of the different groups[7]. It should be clear that when the distributions of the groups do not overlap, OV will be equal to 0.

[7] A similar result was obtained when decomposing the Gini index by population subgroups (see, Silber, 1989).

6. The Bonferroni index and relative deprivation[8]

Using (11) and (13) we may easily derive that

$$I_B = (1/n)\sum_{i=1}^{n} s_i\{n[\sum_{j>i}(1/j) - \sum_{j<i}(1/i)]\} = \sum_{i=1}^{n} s_i\{\sum_{j>i}(1/j) - \sum_{j<i}(1/i)\} \tag{28}$$

We can easily interpret expression (28) in terms of relative deprivation by stating that the Bonferroni index is equal to the income weighted average of the "net deprivation", $\mathrm{ND}(x_i)$, of the various individuals, where

$$\mathrm{ND}(x_i) = [\sum_{j>i}(1/j) - \sum_{j<i}(1/i) \tag{29}$$

In other words the "net deprivation" associated with a given level of income x_i, $\mathrm{ND}(x_i)$, is equal to the difference between the deprivation $\mathrm{D}(x_i)$ and the satisfaction $\mathrm{S}(x_i)$ associated with income x_i, as they are defined below.

The deprivation felt by individual with income x_i with respect to individual with income x_j, is given by

$$\mathrm{D}(x_i, x_j) = \begin{cases} 1/j & \text{if } x_i < x_j \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

For a given individual i, $\mathrm{D}(x_i, x_j)$ is greater, the closer the rank of individual j with respect to that of individual i.

The overall deprivation associated with income x_i, $\mathrm{D}(x_i)$, is then expressed as

$$\mathrm{D}(x_i) = \sum_{j>i}(1/j) \tag{31}$$

so that $\mathrm{D}(x_i)$ will be higher, the poorer individual i is and the marginal increase in individual deprivation is higher, the poorer the individual. On the other hand, the satisfaction felt by individual with income x_i with respect to individual with income x_j, $\mathrm{S}(x_i, x_j)$, is given by:

$$\mathrm{S}(x_i, x_j) = \begin{cases} 1/i & \text{if } x_i > x_j \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

8 Previous interpretations of the Bonferroni index in terms of relative deprivation may be found in Chakravarty (2007) and Bárcena and Imedio (2008).

that is, for a given income x_i, $S(x_i, x_j)$ is constant when the individual compares himself with those poorer than him, and it is zero with respect to those with a higher income.

The overall satisfaction associated with income x_i, $S(x_i)$, is then written as

$$S(x_i) = \sum_{j<i} (1/i) = (i-1)/i \tag{33}$$

Note that this satisfaction $S(x_i)$ is higher, the richer the individual but the marginal increase in this satisfaction is smaller, the richer the individual.[9]

It is then easy to check, combining expressions (29) to (33) that the net deprivation is

$$ND(x_i) = D(x_i) - S(x_i) \tag{34}$$

7. The Bonferroni index and social welfare[10]

Using (11) we can also attempt to derive the social welfare function that lies behind the Bonferroni index. Following Atkinson's (1970) concept of "equally distributed equivalent level of income" we may define the Bonferroni index as

$$I_B = 1 - \frac{x_E^B}{\bar{x}} \tag{35}$$

where $\bar{x}$ is the average income in the population and x_E^B the "equally distributed equivalent level of income" corresponding to Bonferroni index. We then derive, combining (11) and (35), that

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9 It is interesting to notice (see, Berrebi and Silber, 1985) that the Gini index I_G may be also expressed as an income weighted sum of net satisfaction since it may be written as $I_G = \sum_{i=1}^{n} \{((n-i)/n) - ((i-1)/n)\}$ but in this case the incomes are ranked by decreasing values. The gross satisfaction of individual i is then $((n-i)/n)$, that is, the proportion of individuals earning less than individual *i*, while the deprivation is expressed as $((i-1)/n)$, which represents the proportion of individuals earning more than individual *i*. Here also the richer an individual, the higher her gross satisfaction, but the marginal increase remains constant. Note also that this interpretation of the Gini index in terms of income weighted net satisfactions indicates that the poorer the individual, the higher her gross deprivation, but here again the marginal increase in deprivation remains constant.

10 Previous interpretations of the Bonferroni index in terms of social welfare may be found in Chakravarty (2007) and Bárcena and Imedio (2008).

$$x_E^B = \bar{x}(1 - I_B) = \bar{x}(1 - [e'Bs]) = \bar{x}\{[n(e'Is)] - [e'Bs]\} \qquad (36)$$

where I is the identity matrix and use is made of the fact that $(e'Is) = (1/n)$.

Calling N the diagonal matrix whose diagonal elements are all equal to n, we then derive that

$$x_E^B = \bar{x}\{[e'Ns] - [e'Bs]\} = \bar{x}[e'Hs] \qquad (37)$$

where H is a matrix defined as

$$\mathrm{H} = \mathrm{N} - \mathrm{B} \qquad (38)$$

Using (13) we easily find out that H is expressed as

$$\mathrm{H} = \begin{bmatrix} n & -n/2 & -n/3 & \ldots & -n/i & \ldots & -n/(n-1) & -n/n \\ n/2 & n & -n/3 & \ldots & -n/i & \ldots & -n/(n-1) & -n/n \\ n/3 & & n & \ldots & -n/i & \ldots & -n/(n-1) & -n/n \\ \ldots. & & & \ldots & \ldots & \ldots & -n/(n-1) & -n/n \\ n/i & n/i & n/i & \ldots & n & \ldots & -n/(n-1) & -n/n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ n/(n-1) & n/(n-1) & n/(n-1) & \ldots & n/(n-1) & \ldots & n & -n/n \\ n/n & n/n & n/n & \ldots & n/n & \ldots & n/n & n \end{bmatrix} \qquad (39)$$

Remembering that the typical element s_i of the vector s is equal to $(x_i / n\bar{x})$, we then conclude that

$$x_E^B = (1/n)[e'Hx] \qquad (40)$$

where x is the vector of the actual incomes x_i ranked by increasing values.

Combining (39) and (40) we then conclude that x_E^B may be expressed as

$$x_E^B = e'Kx \qquad (41)$$

where K is a matrix defined as

$$K = \begin{bmatrix} 1 & -1/2 & -1/3 & \dots & -1/i & \dots & -1/(n-1) & -1/n \\ 1/2 & 1 & -1/3 & \dots & -1/i & \dots & -1/(n-1) & -1/n \\ 1/3 & & 1 & \dots & -1/i & \dots & -1/(n-1) & -1/n \\ \dots. & & & \dots & \dots & \dots & -1/(n-1) & -1/n \\ 1/i & 1/i & 1/i & \dots & 1 & \dots & -1/(n-1) & -1/n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1/(n-1) & 1/(n-1) & 1/(n-1) & \dots & 1/(n-1) & \dots & 1 & -1/n \\ 1/n & 1/n & 1/n & \dots & 1/n & \dots & 1/n & 1 \end{bmatrix} \quad (42)$$

Combining (41) and (42) we derive that

$$x_E^B = (1/n)\sum_{i=1}^{n} x_i\{[\sum_{j<i}^{n}(-1/i)] + [1] + [\sum_{j>i}(1/j)]\} = \sum_{i=1}^{n} x_i\{(1/n)[\sum_{j<i}^{n}(-1/i) + 1 + \sum_{j>i}(1/j)]\} \quad (43)$$

$$\Leftrightarrow x_E^B = \sum_{i=1}^{n} x_i\{(1/n^2)[\sum_{j>i}(n/j)] + n - \sum_{j<i}^{n}(n/i)]\} \quad (44)$$

It is easy to observe that the sum of the weights in (43) is equal to $(1/n)$ times the sum of the elements of the matrix K. Since the sum of the elements of this matrix K, is equal to n, the sum of the weights in (44) is equal to 1.

8. A Generalization

Let us assume a population of n individuals. We now extend expression (13) and define a "generalized Bonferroni matrix" D as a n by n square matrix whose typical element d_{ij} is equal to 0 if $i = j$, to $(n/j)^{\alpha}$ when $j \succ i$ (with $0 \le \alpha \le 1$) and to $-((n/i)^{\alpha})$ when $i \succ j$, with $0 \le \alpha \le 1$. It is easy to observe that when $\alpha = 1$ the matrix D is identical to the Bonferroni B-matrix defined and used previously. On the other hand when $\alpha = 0$, the matrix D is identical to the G-matrix which is used to compute the Gini-index (see, Silber, 1989). We may therefore define a generalized Bonferroni index B_{GEN} as

$$B_{GEN} = e' Ds \quad (45)$$

Similarly expression (44) may now be generalized and written as

$$\Leftrightarrow x_E^{GEN,B} = \sum_{i=1}^{n} x_i\{(1/n^2)[\sum_{j>i}(n/j)^{\alpha} + n - \sum_{j<i}^{n}(n/i)^{\alpha}]\} \quad (46)$$

But expression (46) may be also written as

$$x_E^{GEN,B} = \sum_{i=1}^{n} w_i x_i \tag{47}$$

with

$$w_i = (1/n^2)\left[\sum_{j>i} a_{ij}(n/j)^{\alpha} + n + \sum_{j<i} a_{ij}(n/i)^{\alpha}\right] \tag{48}$$

where $a_{ij} = -1$ if $1 \leq j < i$, and $a_{ij} = 1$ if $j > i$

Note that when $\alpha = 1$, we obtain the "equally distributed equivalent level of income" corresponding to the Bonferroni index given in (44).

When $\alpha = 0$, since $(n/j)^0 = 1$ and $(n/i)^0 = 1$, we obtain the "equally distributed equivalent level of income" corresponding to the Gini index since such a case (46) may be expressed as

$$x_E^{GEN,B} = (1/n^2)\sum_{i=1}^{n} x_i[(n-i)+n-(i-1)] = (1/n^2)\sum_{i=1}^{n} x_i(2n-2i+1) \tag{49}$$

which corresponds to the "equally distributed equivalent level of income " of the Gini index (see, Donaldson and Weymark, 1980)[11].

Note also that if in (48) $\alpha = 0$ and $a_{ij} = 0$ $\forall j, \forall i, w_i = (1/n)$ $\forall i$, so that $x_E^{GEN,B} = \bar{x}$ and there is no inequality (the "utilitarian" approach).

Finally, when $\alpha = 0$, if in (48) $a_{ij} = 1$ $\forall j$ for all $i \neq 1$ while $a_{1j} = n$ $\forall j$, we will have $w_1 = (1/n^2)(n + n(n-1)) = (n^2/n^2) = 1$ and $w_i = (2n-1)/n^2$ for $i \neq 1$. It is easy to observe that when $n \to \infty$, $w_i \to 0$ for $i \neq 1_i$ while $w_1 = 1$. In such a limit case the "equally distributed equivalent level of income" becomes equal to the income of the poorest individual and we have the Rawlsian case.

11 Obviously if we had ranked the incomes by decreasing values expression (49) would have become $x_E^{GEN,B} = (1/n^2)\sum_{i=1}^{n} x_i(2n - 2(n-i+1)+1) = (1/n^2)\sum_{i=1}^{n} x_i(2i-1)$ which is expression (11) in Donaldson and Weymark (1980).

9. An Empirical Illustration

The empirical investigation is based on the European Union Statistics on Income and Living Conditions (EU-SILC) data set for the 2008 wave. We focus on disposable income in Spain and Sweden. These are two quite different countries since Spain is characterized by a higher degree of inequality in disposable income and a smaller level of state intervention as far as welfare policy is concerned (see Brandolini and Smeeding 2008).

Decomposition of inequality by income sources

Disposable money income includes net income from work, other private income not related to work, pensions and other social transfers. Net money income includes all income sources received by the household and by each of its current members in the year preceding the survey. Social insurance contributions, pay-as-you-earn taxes and non-money income that may be received by the household are not included in this definition of income.

Since a given level of household income will correspond to a different standard of living depending on the size and composition of the household, we adjust for these differences using the modified-OECD equivalence scale[12] [13].

The decomposition of the Bonferroni inequality index by income sources will be based on five income sources:

1. Old-age and survivor's benefits (oldbenefits),
2. Benefits other than old-age and survivor's benefits (otherbenefits),
3. Income from rental of a property or land (property),
4. Interest, dividends, profit from capital investments in unincorporated business (interest),
5. Income available before including sources 1 to 4 (incomebefore)

As mentioned previously we compare results based on the Spanish and Swedish data. It turns out that inequality, measured via the Bonferroni index, is higher in Spain (0.42) than in Sweden (0.33). Table 1 shows the share of each income source in total income. In both countries “incomebefore” represents the largest share in total income (77% in Sweden and almost 81% in Spain). As a whole, benefits have similar shares in total income in both countries, but the share of old age benefits (“oldbenefits”) is substantially higher in Spain. Column 3 in Table 1 gives the degree of inequality of the distribution of the various income sources. In Spain there is more inequality in the distribution of “otherbenefits” and “incomebefore” than in Sweden. The greatest difference in inequality concerns “oldbenefits” since for this source the Bonferroni index is equal to 0.59 in Sweden and to only 0.30 in Spain. But, as stressed

[12] For a survey of equivalence scales and related income distribution issues, and some comparisons of scale relativities, see Coulter et al. (1992).

[13] This scale attaches a value of 1 to the first adult member of the household, 0.5 to the remaining adult members and 0.3 to each member under 14 years of age.

previously, the degree of inequality of the distribution of a given income source does not represent its contribution to overall inequality. The contribution of a given income source to total inequality, as shown in (15), is equal to the product of the share of this source in total income, the Bonferroni index for this source and the "Bonferroni correlation coefficient" for this same source. The contributions of the various income sources are given in column 5 of Table 1.

Table 1: Decomposition of the Bonferroni index by income source in Sweden and Spain

	Share in total income	Inequality (Bonferroni) in each income source	Bonferroni correlation	Share in total inequality
Spain				
oldbenefits	12.1%	0.300	0.680	6.0%
otherbenefits	4.9%	0.310	0.405	1.5%
property	1.5%	0.708	0.782	2.0%
interest	0.8%	0.656	0.868	1.1%
incomebefore	80.6%	0.458	0.997	89.4%
Sweden				
oldbenefits	4.3%	0.587	0.882	6.9%
otherbenefits	14.7%	0.254	-0.470	-5.4%
property	0.1%	0.823	0.511	0.1%
interest	3.5%	0.738	0.945	7.5%
incomebefore	77.4%	0.383	0.998	91.0%

Source: EU-SILC 2008

The components whose contribution to total inequality is higher than their share in total income can be considered as "inequality increasing". This is thus the case for "property", "interest" and "incomebefore" in Spain and Sweden, and also "oldbenefits" in Sweden. We may therefore conclude that in Spain benefits have an equalizing impact while this is true only for "otherbenefits" in Sweden. We may also observe that "otherbenefits" in Sweden have a negative Bonferroni correlation coefficient. This clearly implies that in this country "otherbenefits" are mainly received by low income households.

Decomposition of inequality by income classes

Here we divided the income distribution into five non-overlapping income classes (quintiles) and looked at the data of three countries: Spain, Sweden and Austria.

Table 2: Decomposition of the Bonferroni index by income classes in Sweden and Spain

	Spain	Sweden	Austria
B^{w}	0.0250	0.0209	0.0229
$B^{residual\ \ w}$	0.0185	0.0214	0.0157
B_{BET}	0.3684	0.2829	0.3108
B	0.4119	0.3253	0.3494

Source: EU-SILC 2008

We observe that in all three countries the between-classes Bonferroni index is higher than the within-classes index (see Table 2). Sweden and Austria show less inequality than Spain, for both between and within- income classes inequality. When we compare Austria and Sweden we observe that the former has less within-classes inequality than Sweden, but more between-income inequality.

Decomposition of inequality in the case of overlapping groups

The decomposition of the Bonferroni index by overlapping subgroups is illustrated by classifying incomes in Portugal by degree of urbanization of the area where the individuals live. We consider three degrees of urbanization: densely populated area, intermediate area and thinly populated area.

In this case the results for Portugal are:

$$I_{B,\,TOTAL} = e'Bs = I_{B,TOTAL} = [(e'B\tau)-(e'B\bar{\tau})]+(e'B\bar{\tau})+[(e'Bs)-(e'B\tau)] =$$
$$= B_{WITH} + B_{BET} + OV = 0.1748 + 0.0966 + 0.1832 = 0.2714 + 0.1832 = 0.4546$$

We observe that inequality between individuals living in areas of different degree of urbanization is smaller than inequality within areas with the same degree of urbanization. Inequality between groups accounts for 21.3% of total inequality while inequality within groups accounts for 38.5% and the overlapping component for 40.3%. In this case the overlapping component is the main component of inequality. This obviously implies that in poorer areas there are still individuals with an income which is higher than some of the incomes in richer areas and, conversely, in richer areas there are individuals with an income smaller than some of the incomes in poorer areas. Given the relative size of this overlapping component, we can easily conclude that such a feature is often observed.

9. Concluding comments

This paper proposed a simple algorithm to compute the Bonferroni index of income inequality, using what we have called the "Bonferroni matrix". More precisely we have suggested defining the Bonferroni index as the product of three elements, namely a row vector representing the individual population shares, the Bonferroni matrix, and a column vector representing the individual income shares, the individuals being ranked by increasing incomes.

Such an algorithm greatly simplifies the decomposition of the Bonferroni index by income sources, income classes or population subgroups. The paper also offers a new interpretation of the Bonferroni index in terms of relative deprivation as well as a new formulation of the Bonferroni-related welfare function. In addition, the paper gives a generalization of the Bonferroni index of which the traditional Bonferroni and Gini indices are special cases. The paper ends by a short empirical illustration, based on EU-SILC data for the year 2008. This empirical analysis proves the usefulness of the computation techniques proposed in this paper.

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11. Appendix: The exact formulation of the correction factor C_p

Assume that the first observation of group p is actually the r^{th} observation in the whole population. Then the numerator of C_p may be expressed as

$$[e'_{n_p} B(n_p, n_p) s_{n_p}] =$$

$$= \begin{bmatrix} 1/n_p & \dots & 1/n_p \end{bmatrix} \begin{bmatrix} 0 & n/(r+2) & n/(r+3) & \dots & n/(r+i) & \dots & n/(r+n_p) \\ -n/(r+2) & 0 & n/(r+3) & & n/(r+i) & & n/(r+n_p) \\ -n/(r+3) & -n/(r+3) & 0 & & n/(r+i) & & n/(r+n_p) \\ \dots & & & 0 & & & n/(r+n_p) \\ -n/(r+i) & -n/(r+i) & -n/(r+i) & & & & \dots \\ \dots & \dots & & & & 0 & \\ -n/(r+n_p) & -n/(r+n_p) & -n/(r+n_p) & & & -n/(r+n_p) & 0 \end{bmatrix} \begin{bmatrix} s_1^p \\ s_2^p \\ \\ \\ s_{n_p}^p \end{bmatrix} =$$

$$= (\frac{n}{n_p})[(\frac{-1}{r+2} + \frac{-1}{r+3} + \dots + \frac{-1}{r+n_p})s_1^p + (\frac{1}{r+2} + \frac{-1}{r+3} + \dots + \frac{-1}{r+n_p})s_2^p + \dots + (\frac{1}{r+i}(i-1) + \frac{-1}{r+i+1} + \frac{-1}{r+i+2} + \dots$$
$$+ \frac{-1}{r+n_p})s_i^p + \dots + \frac{1}{r+n_p}(n_p - 1)s_{n_p}^p]$$

whereas the denominator of C_p may be written as

$$[e'_{n_p} B_{n_p} s_{n_p}] =$$

$$= \begin{bmatrix} 1/n_p & \dots & 1/n_p \end{bmatrix} \begin{bmatrix} 0 & n_p/2 & n_p/3 & \dots & n_p/+i & \dots & n_p/n_p \\ -n_p/2 & 0 & n_p/3 & & n_p/+i & & n_p/n_p \\ -n_p/3 & -n_p/3 & 0 & & n_p/+i & & n_p/n_p \\ \dots & & & 0 & & & n_p/n_p \\ -n_p/i & -n_p/i & -n_p/i & & & & \dots \\ \dots & \dots & & & & 0 & \\ -n_p/n_p & -n_p/n_p & -n_p/n_p & & & -n_p/n_p & 0 \end{bmatrix} \begin{bmatrix} s_1^p \\ s_2^p \\ \\ \\ s_{n_p}^p \end{bmatrix} =$$

$$= (\frac{n_p}{n_p})[(\frac{-1}{2} + \frac{-1}{3} + \dots + \frac{-1}{n_p})s_1^p + (\frac{1}{2} + \frac{-1}{3} + \dots + \frac{-1}{n_p})s_2^p + \dots + (\frac{1}{i}(i-1) + \frac{-1}{i+1} + \frac{-1}{i+2} + \dots + \frac{-1}{n_p})s_i^p + \dots + \frac{1}{n_p}(n_p - 1)s_{n_p}^p]$$