OPHI
OXFORD POVERTY & HUMAN DEVELOPMENT INITIATIVE
www.ophi.org.uk

UNIVERSITY OF OXFORD

Summary of Useful Formulas of the AF Method

Maria Emma Santos

CONICET-UNS & OPHI

Achievement Matrix
Cutoff vector & Weights vector

- Where x_{ij} is the achievement of individual i of attribute or dimension j.
- zj is the deprivation cutoff of attribute or dimension j.
- w_j is the weight of attribute or dimension j such that:

$$w_1 + w_2 + \ldots + w_d = \mathbf{d}$$

Dimensions

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \cdots & & \\ & & \cdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}$$

People

$$z = (z_1, z_2, \ldots\ldots, z_d)$$

$$w = (w_1, w_2, \ldots\ldots, w_d)$$

Deprivation Matrix

Where

- $g_{ij}^{0}=1$ if $x_{ij}<z_j$ (deprived)
- $g_{ij}^{0}=0$ if $x_{ij}\geq z_j$ (non-deprived)

- Or equivalently:

$$g_{ij}^{0}=\left(\frac{z_j-x_{ij}}{z_j}\right)^{0}$$

Dimensions

$$g^{0}=\begin{bmatrix} g_{11}^{0} & \cdots & g_{1d}^{0} \\ g_{21}^{0} & \cdots & g_{2d}^{0} \\ \cdots & & \\ & & \cdots \\ g_{n1}^{0} & \cdots & g_{nd}^{0} \end{bmatrix}$$

People

$$z=(z_1,z_2,\ldots\ldots,z_d)$$

Raw Dimensional Headcount Ratios

- These are the deprivation rates by dimension, ie. the proportion of people who are deprived in that dimension.
- It is simply the mean of each column of the deprivation matrix:

$$H_j = (g_{1j}^0 + g_{2j}^0 + \ldots + g_{nj}^0) / n$$

Weighted Deprivation Matrix

Note that we use the same notation as for the deprivation matrix on purpose.

Where

- $g_{ij}^0 = w_j$ if $x_{ij} < z_j$ (deprived)
- $g_{ij}^0 = 0$ if $x_{ij} \geq z_j$ (non-deprived)

- Or equivalently:

$$g_{ij}^0 = w_j \left(\frac{z_j - x_{ij}}{z_j} \right)^0$$

$$g^0 = \begin{bmatrix} g_{11}^0 & \cdots & g_{1d}^0 \\ g_{21}^0 & \cdots & g_{2d}^0 \\ \cdots & & \\ & & \cdots \\ g_{n1}^0 & \cdots & g_{nd}^0 \end{bmatrix}$$

$$\mathrm{z} = (\mathrm{z}_1, \mathrm{z}_2, \ldots\ldots, \mathrm{z}_\mathrm{d})$$

$$w = (w_1, w_2, \ldots\ldots, w_d)$$

Deprivation Count Vector

Where the 'deprivation count' or score for each person is the sum of her weighted deprivations

- $c_i = g_{i1} + \dots + g_{id}$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \\ \\ c_n \end{bmatrix}$$

Identify the poor

Given a poverty cut-off k, we compare the deprivation count with the k cutoff and then censor the deprivations of those who were not identified as poor.

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$\rho_k(x_i; z) = 1 \quad if \quad c_i \geq k \quad poor$$

$$\rho_k(x_i; z) = 0 \quad if \quad c_i < k \quad non-poor$$

Censored Weighted Deprivation Matrix and Deprivation Count Vector

This is the key matrix (and vector, alternatively) over which we compute the set of AF indicators for M_0

$$g^0(k) = \begin{bmatrix} g^0_{11}(k) & \cdots & g^0_{1d}(k) \\ g^0_{21}(k) & \cdots & g^0_{2d}(k) \\ \cdots & & \\ & & \cdots \\ g^0_{n1}(k) & \cdots & g^0_{nd}(k) \end{bmatrix} \qquad c(k) = \begin{bmatrix} c_1(k) \\ c_2(k) \\ \\ \\ c_n(k) \end{bmatrix}$$

Where

- **$g_{ij}^0(k)=g_0$ (that is $=(w_j)$) if $c_i \geq k$ (deprived & poor)**
- **$g_{ij}^0(k)=0$ if $c_i<k$ (deprived or not but non-poor)**
- Similarly: **$c_i(k)=ci$ if $c_i \geq k$ and $c_i(k)=0$ if $c_i<k$**

First we focus on the M_0 measure and all its related indicators

Headcount Ratio of MD Poverty

It is the proportion of people who have been identified as poor. Thus:

$$H = \frac{\sum_{i=1}^{n} \rho_k(x_i; z)}{n} = \frac{q}{n}$$

Where q is the number of poor people.

Headcount Ratio is sometimes called the *incidence* of poverty, or the poverty rate.

Intensity (or breadth) of MD Poverty

- It is the average proportion of deprivations in which the poor are deprived.

$$A = \frac{\sum_{i=1}^{n} c_i(k)}{dq}$$

Note that it is simple to compute:

1) You compute the proportion of total deprivations each poor person has ($c_i(k)/d$). Note we need to use the censored deprivation count vector, ie: we ignore the deprivations of the non-poor.
2) You take the average of those proportions (that's why we divide by q, the number of the poor)

Multidimensional Poverty: M_0

(Adjusted Headcount Ratio)

- It is the product of incidence and intensity.

$$M_0 = H * A$$

- Or equivalently, it is the mean of the censored (weighted) deprivation matrix:

$$M_0 = \mu(g^0(k)) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^0}{nd}$$

How do we interpret M_0?

- M_0 is the mean of the (weighted) censored deprivation matrix, ie: the sum of all the non-zero entries (each weighted by the corresponding indicator weight) divided by the total number of entries (people x indicators).

How do we interpret M_{0}?

- **Weighted Censored Deprivation Matrix:**

(1) Its total number of entries provides the total number of deprivations a society can experience (considered indicators x people).

(2) Its total number of non-zero weighted entries provides the total number of weighted deprivations that the poor actually experience in that society.

How do we interpret M_0?

- M_0 Interpretation

  M_0 is the ratio of (2)/(1), ie. the mean of the weighted censored deprivation matrix.

  Thus, **it gives the proportion of weighted deprivations that the poor experience in a society of all the total potential deprivations that the society could experience**.

How do we decompose M_0 by Indicators and Dimensions?

Visually…

Peo	Ye. Ed	Child Attend.	Nutr	Mor	Elec	Wat	Sani	Floor	Cook. Fuel	Assets
1	1.67	1.67	0	0	0	0.56	0	0.56	0	0
2	0	0	1.67	1.67	0	0	0	0	0	0
3	0	0	0	0	0.56	0.56	0.56	0.56	0.56	0.56
4	1.67	0	0	1.67	0.56	0.00	0.56	0	0.56	0
5	0	0	0	0	0	0	0	0	0	0

What is the contribution of deprivation in health to overall poverty? And in nutrition in particular?

How do we decompose M_0 by Indicators and Dimensions?

There are two useful but distinct indicators to look at:

1) Censored headcount ratios
2) Contributions by indicators and dimensions.

Censored Headcount Ratios

- How do they differ from 'raw' headcount ratios?
- Raw headcount ratios are the % of people who are deprived in a certain indicator.
- **Censored headcount ratios are the % of people who are poor <u>and</u> deprived in a certain indicator.**
- Careful! Censored headcounts **<u>are not</u>** the % of the poor deprived in a certain indicator.

Censored Headcount Ratios

- They are simply the mean of each column of the (weighted) censored deprivation matrix divided by the indicator's weight.

$$H_j^C = \frac{\sum_{i=1}^{n} g_{ij}^0(k)}{w_j n}$$

- Note that M0 is the weighted sum of the censored headcount ratios.

$$M_0 = \sum_{j=1}^{d} \left(\frac{w_j}{d} \right) H_j^C$$

Contribution by Indicator and Dimension

- It is the proportion of total poverty which arises from a particular deprivation.
- Recall from the previous slide:

$$M_0 = \sum_{j=1}^{d} \left(\frac{w_j}{d} \right) H_j^C$$

- Thus, the contribution of indicator j to overall poverty is given by:

$$C_j = \frac{(w_j / d) H_j^C}{M_0}$$

Contribution by Indicator and Dimension

- Note: The sum of the contributions of all d indicators needs to add up to 1 (or 100%).
- Whenever the contribution to poverty of a certain indicator widely exceeds its weight, this suggests that there is a relative high deprivation in this indicator in the country. The poor are more deprived in this indicator than in others.
- If there are more than one indicators in a dimension, the dimensional contribution is simply the sum of the indicators' contribution.

How do we decompose M_0 by population subgroups?

Visually…

GROUP A

Peo	Ye. Ed	Child Attend.	Nutr	Mor	Elec	Wat	Sani	Floor	Cook. Fuel	Assets	Depr. Count
1	1.67	1.67	0	0	0	0.56	0	0.56	0	0	4.44
2	0	0	1.67	1.67	0	0	0	0	0	0	3.33
3	0	0	0	0	0.56	0.56	0.56	0.56	0.56	0.56	3.33
4	1.67	0	0	1.67	0.56	0.00	0.56	0	0.56	0	5
5	0	0	0	0	0	0	0	0	0	0	0

GROUP B

Decomposition by Population Subgroups

If the entire population X (of size n) is divided into two subgroups X_1 (of size n_1) and X_2 (of size n_2), then overall M_0 is the weighted sum of M_0 in each subgroup:

$$M_0(X;z) = \left(\frac{n_1}{n}\right) M_0(X_1;z) + \left(\frac{n_2}{n}\right) M_0(X_2;z)$$

Thus, the contribution of subgroup *i* to overall poverty is

$$C_{Gi} = \frac{(n_i / n) M_0(X_i;z)}{M_0(X;z)}$$

Decomposition by Population Subgroups

- Note that the sum of the contributions of all groups needs to add up to 1 (or 100%).
- Whenever the contribution to poverty of a region or some other group widely exceeds its population share, this suggests that there is a seriously unequal distribution of poverty in the country, with some regions or groups bearing a disproportionate share of poverty.

Generalising the formulas so that they also apply to M_1 and M_2

g^{α} Matrix

$$g^{\alpha} = \begin{bmatrix} g_{11}^{\alpha} & \cdots & g_{1d}^{\alpha} \\ g_{21}^{\alpha} & \cdots & g_{2d}^{\alpha} \\ \cdots & & \\ & & \cdots \\ g_{n1}^{\alpha} & \cdots & g_{nd}^{\alpha} \end{bmatrix}$$

Where

$$g_{ij}^{\alpha} = w_j \left(\frac{z_j - x_{ij}}{z_j} \right)^{\alpha} \quad if \quad x_{ij} < z_j$$

$$g_{ij}^{\alpha} = 0 \quad if \quad x_{ij} \geq z_j$$

Censored g^{α} Matrix

(after identification, done exactly the same as with M_0)

$$g^{\alpha}(k) = \begin{bmatrix} g_{11}^{\alpha}(k) & \dots & g_{1d}^{\alpha}(k) \\ g_{21}^{\alpha}(k) & \dots & g_{2d}^{\alpha}(k) \\ \dots & & \\ & & \dots \\ g_{n1}^{\alpha}(k) & \dots & g_{nd}^{\alpha}(k) \end{bmatrix}$$

Where

$$g_{ij}^{\alpha}(k) = g_{ij}^{\alpha} \quad if \quad c_i \geq k$$

$$g_{ij}^{\alpha}(k) = 0 \quad if \quad c_i < k$$

Multidimensional Poverty in general
M_1: Adjusted Poverty Gap
M_2: Adjusted FGT

$$M_\alpha = \mu(g^\alpha(k)) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{d} g_{ij}^\alpha}{nd}$$

Note that M_1: Adjusted Poverty Gap

$$M_1 = H * A * G$$

Where G is the average poverty gap across all instances in which poor people are deprived

$$G = \frac{\sum_{i=1}^{n}\sum_{j=1}^{d} g_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{d} g_{ij}^{0}}$$

Note that M_2: Adjusted FGT

$$M_2 = H * A * S$$

Where S is the average squared poverty gap (average severity of deprivations) across all instances in which poor people are deprived

$$S = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^{2}}{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^{0}}$$

Contribution by Indicator

- It is the proportion of total poverty which arises from a particular deprivation.
- In general:

$$M_\alpha = \sum_{j=1}^{d} \mu(g_{*j}^{\alpha}(k)) / d$$

- Where:

$$\mu(g_{*j}^{\alpha}(k)) = \sum_{i=1}^{n} g_{ij}^{\alpha}(k) / n$$

- Thus, the contribution of indicator j to overall poverty is given by:

$$C_j = \frac{\mu(g_{*j}^{\alpha}(k))}{dM_\alpha}$$

Contribution by Population Subgroup

- It is the proportion of total poverty which arises from a particular group.
- In general: $M_\alpha = \sum_{i=1}^{n} \mu(g_i^\alpha(k)) / n$
- For groups: If the entire population X (of size n) is divided into two subgroups X_1 (of size n_1) and X_2 (of size n_2), then overall M_0 is the weighted sum of M_0 in each subgroup

$$M_\alpha(X;z) = \left(\frac{n_1}{n}\right) M_\alpha(X_1;z) + \left(\frac{n_2}{n}\right) M_\alpha(X_2;z)$$

- Thus, the contribution of subgroup *i* to overall poverty is

$$C_{Gi} = \frac{(n_i / n) M_\alpha(X_i;z)}{M_\alpha(X;z)}$$