Summer School on Capability and Multidimensional Poverty

28 August-9 September, 2008

New Delhi, India

HDCP-IRC

The Human Development, Capability and International Research Centre
Instituto Universitario di Studi Superiori
www.iusspavia.it

OPHI

Oxford Poverty & Human Development Initiative
University of Oxford
www.ophi.org.uk

Robustness of Single-dimensional Inequality and Poverty Measures

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30 August, 2008

Introduction

- What is a Robust Comparison?
  - When a particular comparison is unambiguous

- Comparison of what?
  - Inequality
  - Poverty

- Focus of this lecture
  - To learn when any two distributions are comparable and to what extent

Robust Inequality Comparison

Two Classes of Measures

- Class of Lorenz Consistent Inequality Measures

- Class of Transfer Sensitive Inequality Measures

Lorenz Consistent Measures

- Any relative inequality measure satisfying four properties – *Symmetry*, *Replication Invariance*, *Scale Invariance* and *Pigou-Dalton Transfer Principle*, is Lorenz Consistent.

Comparing Two Distributions

- Example: Suppose there are two distributions

  $X = \{10, 20, 30, 40\}$ and $Y = \{10, 25, 25, 40\}$

- Both Distributions have the same mean.

- Which distribution is more equal?

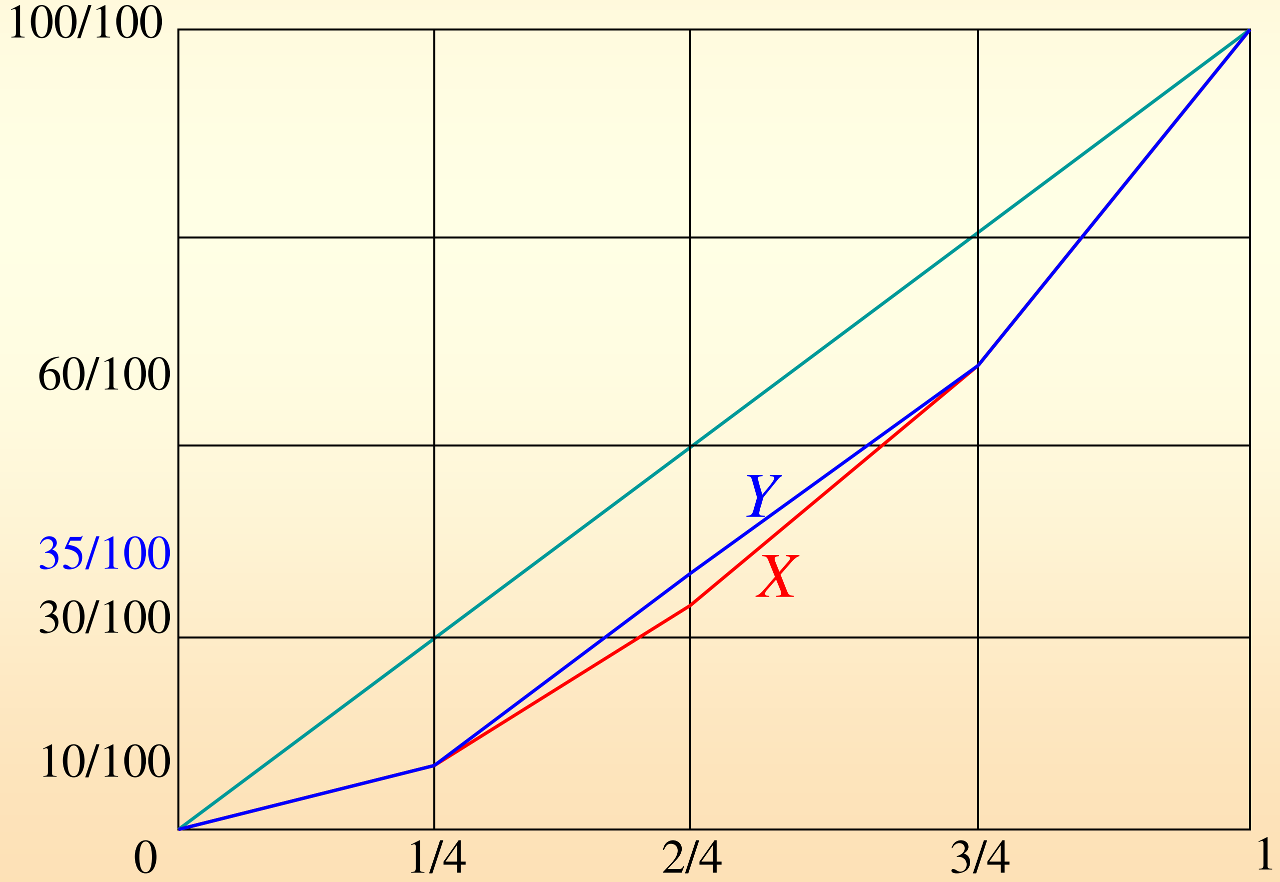
Comparing Two Distributions

- How do the Lorenz curves look?

%Cum. Population (p)	%Cum. Income $L(X, p)$	%Cum. Income $L(Y, p)$
1/4	10/100	10/100
2/4	30/100	35/100
3/4	60/100	60/100
1	100/100	100/100

Comparing Two Distributions

- How the Lorenz curves would look like?



Comparing Two Distributions

- Any relative inequality measure will judge *y* as more equal compared to *x*.

- Comparison of *x* and *y* is completely robust for all relative inequality measures.

- Consider the following pairs of distributions

$$X = \{10, 20, 30, 40\} \text{ and } Y' = \{17, 17, 17, 49\}$$

$$X = \{10, 20, 30, 40\} \text{ and } Y'' = \{19, 19, 19, 43\}$$

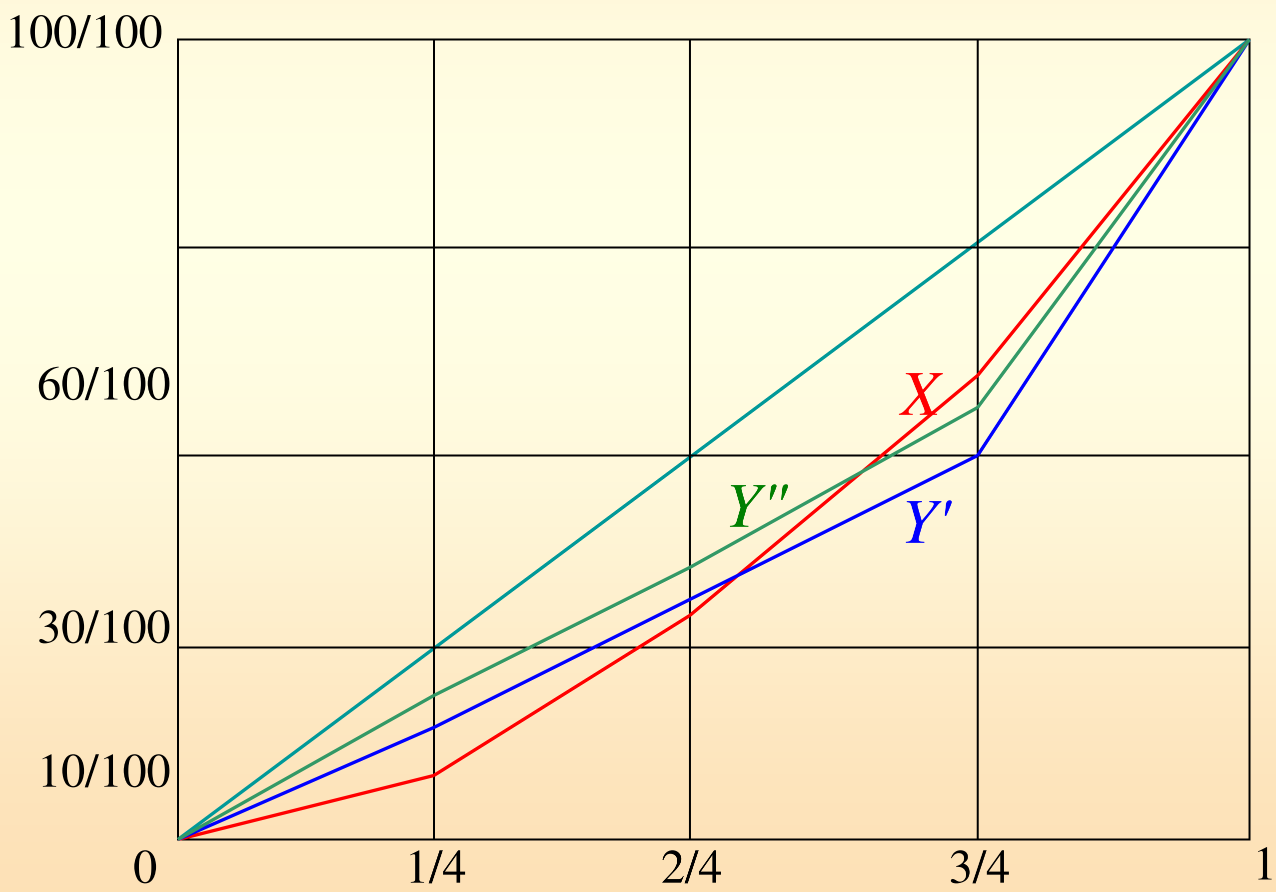
Comparing Two Distributions

- How do the Lorenz curves look?

%Cum. Population (p)	%Cum. Income $L(X, p)$	%Cum. Income $L(Y', p)$	%Cum. Income $L(Y'', p)$
1/4	10/100	17/100	19/100
2/4	30/100	34/100	38/100
3/4	60/100	51/100	57/100
1	100/100	100/100	100/100

Comparing Two Distributions

- How the Lorenz curves would look like?



Which distribution is more equal: X or Y'?

Gini(X) = 0.25
Gini(Y') = 0.24

$CV^2(X) = 0.27$
$CV^2(Y') = 0.41$

Comparison is not robust

Transfer Sensitive Measures

Theorem. (Shorrocks & Foster 1987) Suppose X and Y have the same mean and the Lorenz curve of Y intersects that of X once from above. Then

$I(X) > I(Y)$ for all transfer sensitive inequality measures $I(\cdot)$ if and only if

$$\mathrm{sd}^2(X) \geq \mathrm{sd}^2(Y)$$

Transfer Sensitive Measures

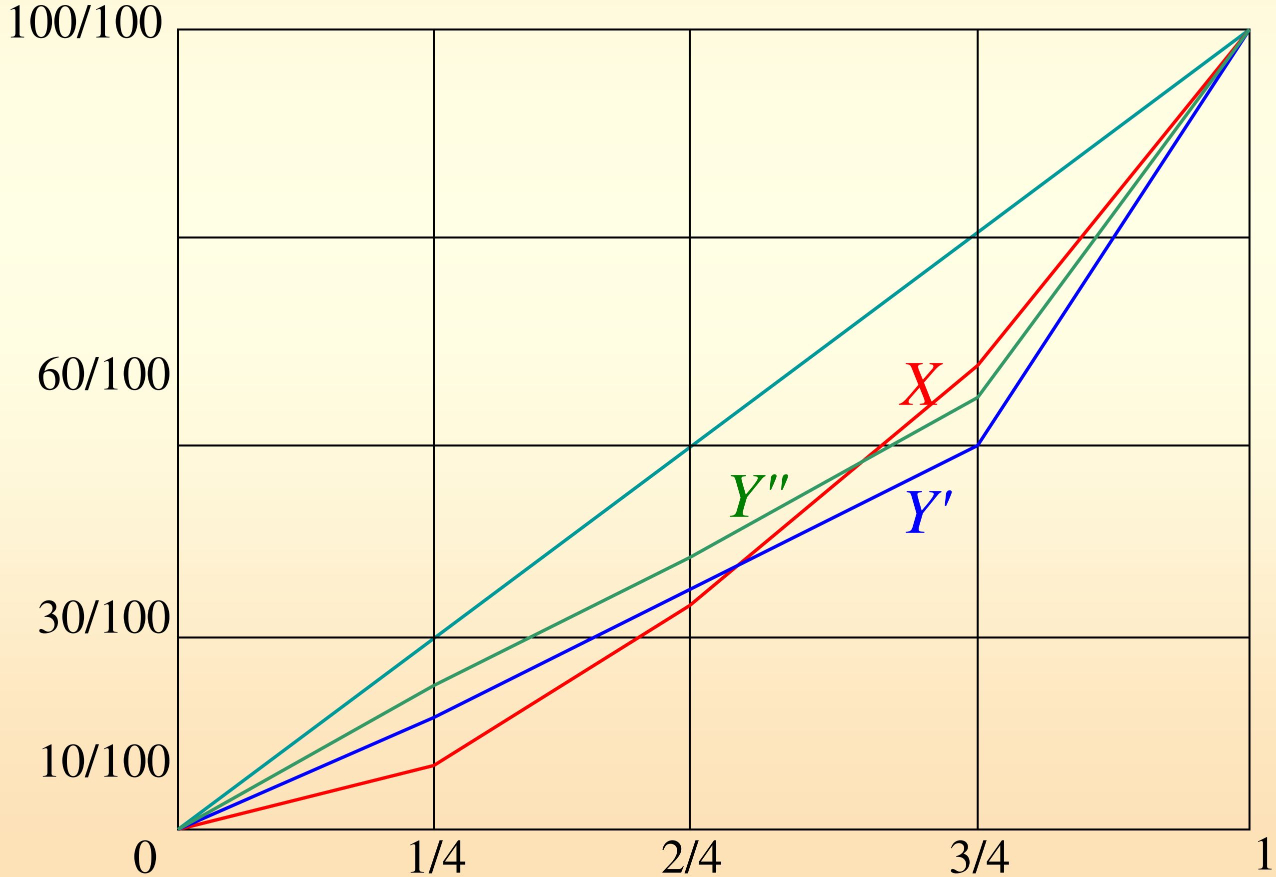
Corollary. (Shorrocks & Foster 1987) Suppose X and Y have positive mean and the Lorenz curve of Y intersects that of X once from above. Then

$I(X) > I(Y)$ for all inequality measures $I(\cdot)$ satisfying transfer sensitivity, scale invariance and replication invariance, if and only if

$$\mathrm{CV}^2(X) \geq \mathrm{CV}^2(Y)$$

Transfer Sensitive Measures

- Which distribution is more equal according to all transfer sensitive inequality indices?



CV(*X*) = 0.27
CV(*Y'*) = 0.41
CV(*Y''*) = 0.23

Y'' is more equal than *X*.

Transfer Sensitive Measures

- Comparison of X and Y'' is robust for all transfer sensitive inequality indices

- Comparison of X and Y' is still not robust

- Note: with same mean, Lorenz consistency is equivalent to *second* order stochastic dominance and Transfer sensitivity is equivalent to *third* order stochastic dominance

Robust Poverty Comparison

Robust Comparison

- When a particular poverty comparison is robust or unambiguous?
- Robust with respect to what?
- Robust with respect to the poverty line chosen.

FGT class of indices

- For this part, we focus on FGT class of indices:

$$\mathrm{FGT}_{\alpha}(x;z)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{z-x_i^*}{z}\right)^{\alpha}=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{g_i^*}{z}\right)^{\alpha}$$

- When $\alpha = 0$, $\mathrm{FGT}_0 = \mathrm{H}$ (Headcount Ratio)
- When $\alpha = 1$, $\mathrm{FGT}_1 = \mathrm{HI} = \mathrm{P}_1$ (Per Capita Income Gap)
- When $\alpha = 2$, $\mathrm{FGT}_2 = \mathrm{P}_2$ (Per Capita Squared Poverty Gap)

Head Count Ratio

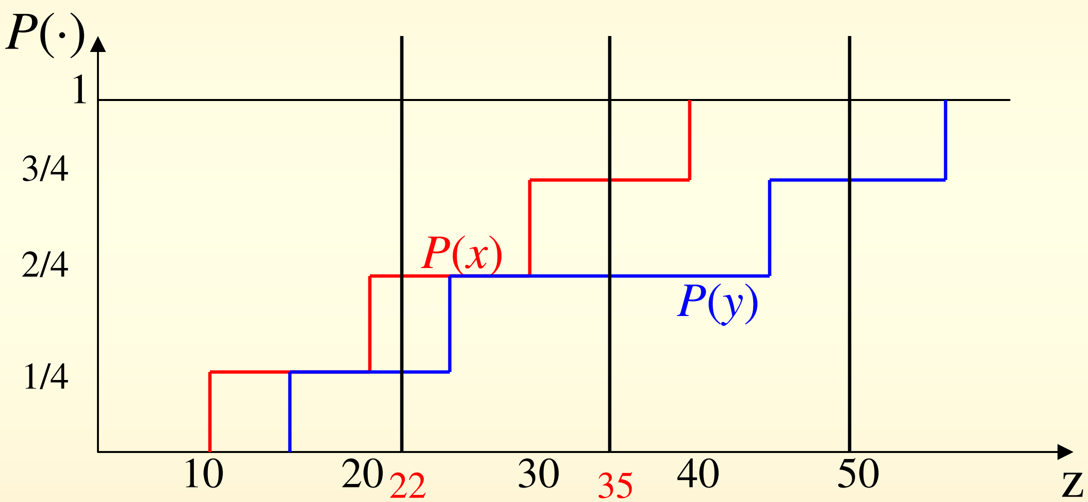
- Head Count Ratio: H = q/n.
- Given income distributions to be compared

$X = \{10, 20, 30, 40\}$, $Y = \{15, 25, 45, 55\}$

- Suppose the poverty line is 22
- H(X; 22) = 2/4, H(Y; 22) = 1/4

Head Count Ratio

- Probability distribution of income $P(\cdot)$



$X = \{10, 20, 30, 40\}$

$Y = \{15, 25, 45, 55\}$

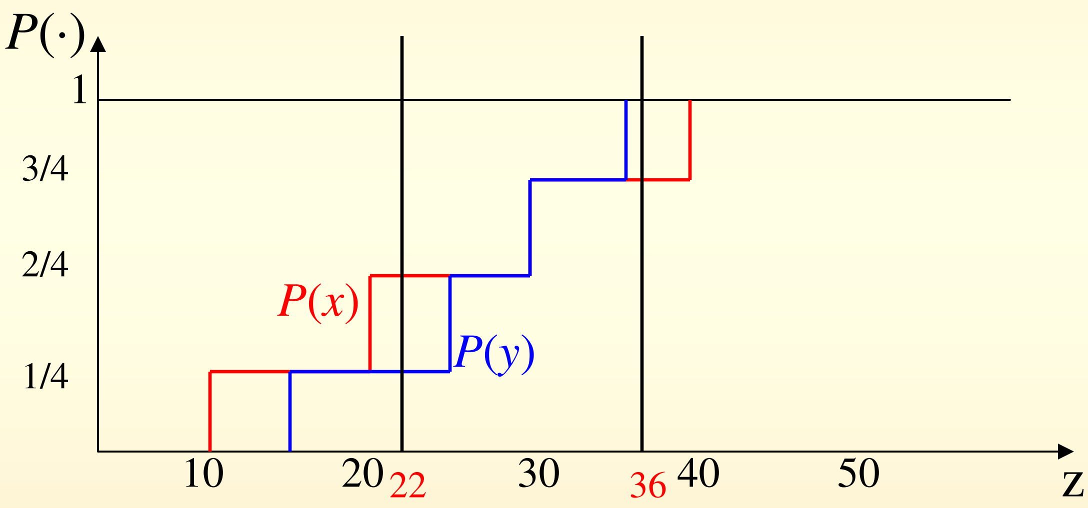
- What happens if poverty line is 35 or 50?
- H(X; 35) = 3/4, H(Y; 35) = 2/4;
- H(X; 50) = 4/4, H(Y; 50) = 3/4

Head Count Ratio

- In terms of head count ratio Y has no greater poverty than X has for all z
- In terms of probability distribution, $P(y)$ FSD $P(x)$ since $P(y \leq z) \leq P(x \leq z)$ $\forall z$ and < for some z
- (Foster & Shorrocks 1988) The poverty ordering H is thus identical to FSD.
- The poverty dominance of Y over X is robust for all poverty lines z if and only if $P(y)$ FSD $P(x)$

Head Count Ratio (not robust)

- Suppose distribution Y becomes {15, 25, 30, 35}



X = {10, 20, 30, 40}

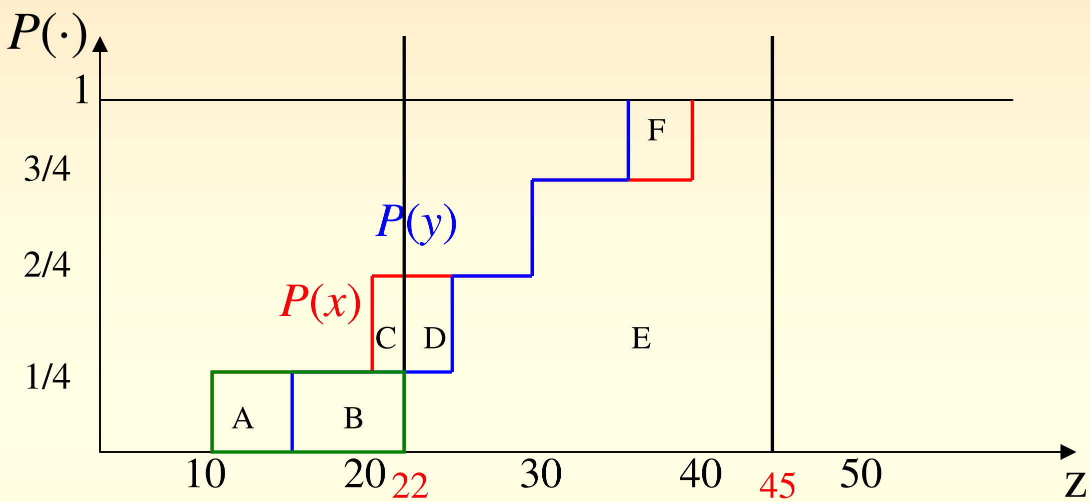
This comparison is not robust for H

- What happens if poverty line is 22 or 36?
- H(X; 22) = 2/4, H(Y; 22) = 1/4;
- H(X; 36) = 3/4, H(Y; 36) = 4/4

Per Capita Income Gap

- Per Capita Income Gap: $P_1(X) = \sum_{i \in q(X)} (z - x_i)/nz$
- Given income distributions to be compared
  $X = \{10, 20, 30, 40\}$, $Y = \{15, 25, 30, 35\}$
- Suppose the poverty line is 22
- $P_1(X; 22) = [(22\text{-}10) + (22\text{-}20)]/(4\times22) = 0.16$
- $P_1\ (Y; 22) = (22\text{-}15)/(4\times22) = 0.08$

Per Capita Income Gap



$(z - x_i)/nz$

- $P_1(X; 22) = (A + B + C)/22$, $I(Y; 22) = B/22$

- If the poverty line is $z = 45$
- $P_1(X; 22) = (A + B + C + D + E + F)/22 = 0.44$
- $P_1(Y; 22) = (B + E + F)/22 = 0.42$

Per Capita Income Gap

- In terms of per capita income gap, *Y* has no greater poverty than *X* for all *z*

- In terms of probability distribution, $P(y)$ SSD $P(x)$ since $\int^{z} P(y \leq s)\mathrm{d}s \leq \int^{z} P(x \leq s)\mathrm{d}s\ \forall z$ and < for some *z*.

- (Foster & Shorrocks 1988) The poverty ordering P_1 is thus identical to SSD.

- Dominance of *Y* over *X* is robust in terms of per capita income gap if and only if $P(y)$ SSD $P(x)$

Per Capita Squared Income Gap

- Per Capita Income Gap is not sensitive to transfer
- Per Capita Squared Income Gap is defined by $P_2(X) = \sum_{i \in q(X)} (z - x_i)^2 / nz^2$
- In terms of probability distribution, $P(y)$ TSD $P(x)$ if and only if $\int^{z}\int^{t} P(y \leq s)\,\mathrm{d}s\,\mathrm{d}t \leq \int^{z}\int^{t} P(x \leq s)\,\mathrm{d}s\,\mathrm{d}t$ $\forall z$ and < for some z.

Per Capita Squared Income Gap

- (Foster & Shorrocks 1988) The poverty ordering P_2 is identical to TSD.

- Dominance of *Y* over *X* is robust in terms of per capita squared income gap if and only if $P(y)$ TSD $P(x)$

Stochastic Dominance Rules

- An easy way to check different degrees of stochastic dominance
- Vectors: $X = \{10, 20, 30, 40\}$, $Y = \{15, 25, 45, 55\}$
- First, arrange them in ascending order (if they are not already)

Stochastic Dominance Rules

- First Degree
  - Check each element of the vectors
  - If $y_i \geq x_i$ for all i, then Y FSD X
- Second Degree
  - Construct X' and Y' such that $x'_1 = x_1$, $y'_1 = y_1$, and $x'_i = x_i + x_{i-1}$, $y'_i = y_i + y_{i-1}$ for $i = 2,3,\ldots$

Stochastic Dominance Rules

- Second Degree
  - $X = \{10, 20, 30, 40\}$, $Y = \{15, 25, 45, 55\}$
  - $X' = \{10, 30, 60, 100\}$ and $Y' = \{15, 40, 85, 140\}$
  - If $y'_i \geq x'_i$ for all i, then Y SSD X
- Third Degree
  - Construct X'' and Y'' such that $x''_1 = x'_1$, $y''_1 = y'_1$, and $x''_i = x'_i + x'_{i-1}$, $y''_i = y'_i + y'_{i-1}$ for $i = 2,3,\ldots$

Stochastic Dominance Rules

- Third Degree
  - $X' = \{10, 30, 60, 100\}$ and $Y' = \{15, 40, 85, 140\}$
  - $X'' = \{10, 40, 100, 200\}$ and $Y'' = \{15, 55, 140, 280\}$
  - If $y''_i \geq x''_i$ for all i, then Y TSD X