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UNIVERSITY OF OXFORD

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Properties of Multidimensional Poverty Measures

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Focus of This Lecture

Discuss the axiomatic structure considered as ‘desirable’ or ‘convenient’ for the measurement of poverty in the multidimensional context

Main Sources of this Lecture

- Bourguignon and Chakravarty. (2003): The Measurement of Multidimensional Poverty
- Alkire and Foster (2007, 2011): Counting and Multidimensional Poverty Measurement
- Please see the reading list for others

Preliminaries

Preliminaries

Multiple dimensions

- Standard of living, knowledge, quality of health (referred as 'achievements')

Achievements of a society or country can be represented by a matrix or joint distribution

Unit of analysis may be individual or household

Preliminaries

A typical dataset or achievement matrix with 4 dimensions

$$
\mathbf{x} = \begin{array}{|cccc|l}
\text{Income} & \text{Years of Education} & \text{Housing Index} & \text{Mal-nourished} & \\
700 & 14 & 4 & \text{No} & \textbf{Person 1} \\
300 & 13 & 5 & \text{Yes} & \textbf{Person 2} \\
400 & 10 & 1 & \text{Yes} & \textbf{Person 3} \\
800 & 11 & 3 & \text{No} & \textbf{Person 4}
\end{array}
$$

$$
\mathbf{z} = \begin{array}{|cccc|}
\mathbf{500} & \mathbf{12} & \mathbf{3} & \textbf{No}
\end{array}
$$

z is the vector of poverty lines

Preliminaries

Matrix $x=[x_{ij}]_{n\times d}$ summarizes the joint distribution of d attributes across n individuals

Row vector $x_{i\bullet}$ denotes the achievements of person i in all d dimensions

Column vector $x_{\bullet j}$ denotes the achievements of all n persons in dimension d

Vector $z=[z_1,...,z_d]$ be the cut-off vector containing the poverty line of each dimension

Preliminaries

A general achievement matrix

x_{ij}: the achievement of individual i in dimension j

Example:

x_{1d}: the achievement of the 1st individual in dimension d

x_{n1}: the achievement of the nth individual in the first dimension

Dimensions

$$x = \begin{bmatrix} x_{11} & \ldots & x_{1d} \\ x_{21} & \ldots & x_{2d} \\ \ldots & & \\ & & \ldots \\ x_{n1} & \ldots & x_{nd} \end{bmatrix} = \begin{bmatrix} x_{1\bullet} \\ x_{2\bullet} \\ \\ \\ x_{n\bullet} \end{bmatrix}$$

Persons

$$\begin{array}{c} \parallel \\ [\, x_{\bullet 1} \quad \ldots \quad x_{\bullet d} \,] \end{array}$$

Multidimensional Poverty Measurement

Measurement

Measurement of multidimensional poverty involves two major steps like unidimensional measurement

- Identification
- Aggregation

First Step: Identification

Identification: *Who is multidimensionally poor*?

An '*identification function*', ρ, decides who should be multidimensionally poor

$\rho(x_{i\bullet},z) = 1$ if person i is multidimensionally poor
$\rho(x_{i\bullet},z) = 0$ if person i is not multidimensionally poor

Unlike the unidimensional framework, there can be two types of identification procedures

Counting Approach
Aggregate Poverty Line Approach

First Step: Identification

Identification: Counting Approach (Two stages)

First stage: Determine whether individuals are deprived in each dimension

Second stage: Identify if someone is poor based on an identification function (criterion)

Three types:

Union criterion (if deprived in at least one dimension)

Intersection criterion (if deprived in all dimensions)

Intermediate criterion

First Step: Identification

Example: Constructing first stage ‘Deprivation Matrix’

$$\mathbf{x} = \begin{array}{|cccc|l} \text{Income} & \text{Years of Education} & \text{Housing Index} & \text{Mal-nourished} & \\ 700 & 14 & 4 & \text{No} & \textbf{Person 1} \\ 300 & 13 & 5 & \text{Yes} & \textbf{Person 2} \\ 400 & 10 & 1 & \text{Yes} & \textbf{Person 3} \\ 800 & 11 & 3 & \text{No} & \textbf{Person 4} \end{array}$$

$$\mathbf{z} = \begin{array}{|cccc|} \mathbf{500} & \mathbf{12} & \mathbf{3} & \textbf{No} \end{array}$$

First Step: Identification

Example: Constructing first stage 'Deprivation Matrix'

Replace entries: 1 if deprived, 0 if not deprived

$$g^0 = \begin{array}{|cccc|l} \text{Income} & \text{Years of Education} & \text{Housing Index} & \text{Mal-nourished} & \\ 0 & 0 & 0 & 0 & \textbf{Person 1} \\ 1 & 0 & 0 & 1 & \textbf{Person 2} \\ 1 & 1 & 1 & 1 & \textbf{Person 3} \\ 0 & 1 & 0 & 0 & \textbf{Person 4} \end{array}$$

$$z = \begin{vmatrix} \mathbf{500} & \mathbf{12} & \mathbf{3} & \textbf{No} \end{vmatrix}$$

These entries fall below cutoffs

First Step: Identification

Identification: Aggregate Poverty Line Approach

A person is identified as poor if her aggregate achievement falls below an aggregate poverty line

Let the aggregation function be denoted by ϕ

Then,

$$\rho(x_{i\bullet},z) = 1 \quad \text{if } \phi(x_{i\bullet}) < \phi(z)$$
$$\rho(x_{i\bullet},z) = 0 \quad \text{if } \phi(x_{i\bullet}) \geq \phi(z)$$

Example consumer expenditure approach

Note: No deprivation matrix was created in this situation

Second Step: Aggregation

Aggregation: *How poor is the society?*

Based on the identification criterion, this step construct an index of poverty P(x;z) summarizing the information of the poor (*a censored matrix can be created just as in the unidimensional framework*)

Axioms

Axioms in Multidimensional Context

Two types

1. *Natural extensions* of the unidimensional framework.

2. Axioms specific to the multidimensional context

Natural Extensions

Symmetry *(Anonymity):* If matrix y is obtained from matrix x by a *permutation* of achievements and the poverty lines remain unchanged, then P(y;z) = P(x;z)

y is obtained from x by a ***permutation*** of incomes if x = Py, where P is a permutation matrix.

Example: $y = Px = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 4 & 4 & 2 \\ 8 & 6 & 3 \end{bmatrix}$

Natural Extensions

***Replication Invariance** (Population Principle)*: If matrix y is obtained from matrix x by a *replication* and the poverty lines remain unchanged, then P(y;z) = P(x;z)

y is obtained from x by a ***replication*** if each person's achievement vector in x is simply repeated a finite number of times

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$ $y = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \\ 8 & 6 & 3 \end{bmatrix}$

Natural Extensions

***Scale Invariance** (Homogeneity of Zero-Degree)*: If all achievements in matrix x and all poverty lines in z are changed by the same *proportion* α > 0, then P(αx;αz) = P(x;z).

Example:

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad z = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix}$$

$$\alpha X = \begin{bmatrix} 2(4) & 2(4) & 2(2) \\ 2(3) & 2(5) & 2(4) \\ 2(8) & 2(6) & 2(3) \end{bmatrix} \qquad \alpha z = \begin{bmatrix} 2(4) & 2(5) & 2(3) \end{bmatrix}$$

Natural Extensions

***Focus*:** Unlike in the unidimensional framework, there are two types of focus axiom

(*Type I*) Focus on those identified as multidimensionally poor' (*we are not interested in those who are not multidimensionally poor*)

(Type II) Focus on dimensions where multidimensionally poor are deprived (*we are not interested in dimensions in which they are not deprived*)

Natural Extensions

Poverty Focus (Type I): If y is obtained from x by *an increment to a non-poor person's achievements* and the poverty lines remain unchanged, then P(y;z) = P(x;z)

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 4 \end{bmatrix}$, $z = (5,6,4)$, and $g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Person 3 is not multidimensionally poor, does it matter if he/she experiences an increase in any of the dimensions?

Natural Extensions

Deprivation Focus (Type II): If y is obtained from x by *a increment in achievements among the non-deprived*, then P(X;z)=P(Y;z). [Recall Deprived vs. Poor]

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 4 \end{bmatrix}$, $z = (5,6,4)$, and $g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Suppose person 2 is considered multidimensionally poor, does it matter if he/she experiences an increment in the third dimension in which he/she is not deprived?

Natural Extensions

Focus Axioms and Types of Identification

Each of the two focus axioms is attributed to a each identification technique introduced earlier

- Poverty focus is attributed to the Aggregated Poverty Line Approach
- Deprivation focus is attributed to the Counting Approach

Natural Extensions

Continuity: For any sequence x , if x' converges to x, then $P(x';z)$ converges to $P(x;z)$

A technical assumption. It prevents poverty measures from changing abruptly for changes in distribution of achievements

Similar intuitive interpretation as the assumption in single dimensional framework

Natural Extensions

Monotonicity: There are, unlike in unidimensional framework, two types of monotonicity axiom

(*Type I*) Becoming less deprived in a specific dimension (within dimension):

(Type II) Becoming deprived in one less dimension (across dimensions): Dimensional Monotonicity

Natural Extensions

Monotonicity: If y is obtained from x by a *deprived increment* among the poor and the poverty line remains unchanged, then P(y,z) < P(x,z)

y is obtained from x by a *deprived increment* if there is an increment in a deprived achievement of a multidimensionally poor

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}, z = (5\ \ 6\ \ 4), y = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$

Person 1 is multidimensionally poor, and experiences an improvement in the third dimension.

Natural Extensions

Dimensional Monotonicity: If y is obtained from x by a *dimensional increment among the poor*, then P(y,z)<P(x,z)

y is obtained from x by a *dimensional increment among the poor* if due to an increment in a deprived achievement of a poor, he or she becomes non-deprived in that dimension

Example: $$x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}, z = (5 \;\; 6 \;\; 4), y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 6 & 4 \\ 8 & 6 & 3 \end{bmatrix}$$

Suppose person 2 is considered multidimensionally poor, and experiences an increment in the second dimension and is no longer deprived in it

Natural Extensions

Population Subgroups

Suppose the population size of x is denoted by n(x). Matrix x is divided into two population subgroups: x' with population size n(x') and x'' with population size n(x'') such that n(x) = n(x') + n(x'')

$$x = \begin{array}{cccc} \textbf{Income} & \textbf{Education} & \textbf{Health} & \\ 4 & 4 & 2 & \textbf{Person 1} \\ 3 & 5 & 4 & \textbf{Person 2} \\ 8 & 6 & 3 & \textbf{Person 3} \end{array}$$

Natural Extensions

Population Subgroup Consistency: If P(y';z) > P(x';z) and P(y'';z) = P(x'';z), and n(x') = n(y'), n(y'') = n(x''), then P(y;z) > P(x;z)

Population Subgroup Decomposability: A poverty measure is additive decomposable if:

$$P(x) = \frac{n(x')}{n} P(x') + \frac{n(x'')}{n} P(x'')$$

Recall: *decomposability implies subgroup consistency, but the converse does not hold*

Analogous Concept: Dimensional Subgroups

$$x = \begin{array}{cccc} \text{Income} & \text{Education} & \text{Health} & \\ 4 & 4 & 2 & \text{Person 1} \\ 3 & 5 & 4 & \text{Person 2} \\ 8 & 6 & 3 & \text{Person 3} \end{array}$$

Decomposability Across Dimensions

It is a purely multidimensional concept, where the overall poverty can be expressed as an weighted average of dimensional deprivations (among poor only)

Natural Extensions

Transfer in unidimensional context: If y is obtained from x by a progressive transfer among the poor, then P(y;z) < P(x;z)

Recall if income is transferred from a person to another who is not richer than the former, keeping mean income same, the transfer is called a *progressive transfer*

This is also known as *Pigou-Dalton* transfer principle

Example: z = 10, x = (9,4,15,8); y = (9,5,15,7)

Natural Extensions

Transfer in multidimensional context:

Bistochastic matrix (B): A matrix whose row elements and column element sum up to one

Example: A general bistochastic matrix $\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$

Multiply a vector by a bistochastic matrix

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 16 \end{bmatrix} = \begin{bmatrix} 7.6 \\ 8.8 \\ 11.6 \end{bmatrix}$$

Natural Extensions

Transfer in multidimensional context:

Bistochastic matrix (B): A matrix whose row elements and column element sum up to one

Example: What bostochastic matrix is used to obtain y = (9,5,15,7) from x = (9,4,15,8)?

It is $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0.25 \\ 0 & 0 & 1 & 0 \\ 0 & 0.25 & 0 & 0.75 \end{bmatrix}$

Natural Extensions

***Uniform Majorization (UM)*:** Matrix y is obtained from x by a *Uniform Majorization among the poor* (an averaging of achievements among the poor) if y = Bx, where B is an n×n bistochastic matrix but not a permutation matrix, and $b_{ii}=1$ for every non-poor person *i* in Y.

$$X = BY = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3.5 & 4.5 & 3 \\ 3.4 & 4.5 & 3 \\ 8 & 6 & 3 \end{bmatrix}, \text{and } z = [5 \;\; 6 \;\; 5]$$

Achievements of the first two persons (poor) were smoothed

Natural Extensions

Transfer Under UM: If y is obtained from x by a *uniform majorization among the poor* (an averaging of achievements among the poor), then P(y;z) ≤ P(x;z).

Axiom Specific to the Multidimensional Case

Consider the following two matrices

$$x = \begin{array}{c} \\ \left[\begin{array}{ccc} \text{Income} & \text{Education} & \text{Health} \\ 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{array}\right] \end{array} \begin{array}{l} \\ \textbf{Person 1} \\ \textbf{Person 2} \\ \textbf{Person 3} \end{array} \qquad y = \begin{array}{c} \\ \left[\begin{array}{ccc} \text{Income} & \text{Education} & \text{Health} \\ 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{array}\right] \end{array} \begin{array}{l} \\ \textbf{Person 1} \\ \textbf{Person 2} \\ \textbf{Person 3} \end{array}$$

$$z = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix}$$

Is the pattern of poverty same in both societies?

If not, what is the difference?

Axiom Specific to the Multidimensional Case

Both matrices have the same distribution for each dimension (*marginal distribution*)

The correlation between dimensions are not same

Require an axiom based on correlation between dimension when marginals are same (Atkinson & Bourguignon, 1982; Boland & Proschan, 1988).

This axiom is intrinsic to the multivariate case

Axiom Specific to the Multidimensional Case

$$x = \begin{bmatrix} 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{bmatrix} \qquad y = \begin{bmatrix} 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{bmatrix}$$

Ways to call the data transformation:

From x to y: *association increasing rearrangement* (Boland & Proschan, 1988); correlation-increasing transfer (Tsui, 1999), correlation increasing switch (Bourguignon & Chakravarty, 2003 and Chakravarty, 2010)

From y to x: association decreasing rearrangement, (Alkire & Foster, 2007, 2011).

Axiom Specific to the Multidimensional Case

Matrix x is obtained from y by an *association decreasing rearrangement among the poor* if for two persons i and i',

i) Person i and person i' are poor in y

ii) In y, in **no** dimension, person i' has more achievement than person i

iii) In x, i and i' switch some of their achievements in such a way that i has more in some dimension and i' has more in some other dimensions (marginal distributions remain same)

iii) $y_{i''\cdot} = x_{i''\cdot}$ for all except i and i' or the amount of attributes of all other persons $i'' \neq i,i'$ remain unchanged

iv) Thus, $y_{i\cdot}$ and $y_{i'\cdot}$ are comparable by vector dominance but not $x_{i\cdot}$ and $x_{i'\cdot}$

Axiom Specific to the Multidimensional Case

Vector Dominance

Vector dominance between vector b = $(b_1, \ldots, b_d)$ and vector c = $(c_1, \ldots, c_d)$ occurs when $b_i \geq c_i$ for all i and $b_i > c_i$ for some I

Vector dominance in y for all three rows, but not in x

$$x = \begin{bmatrix} 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{bmatrix} \quad y = \begin{bmatrix} 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{bmatrix}$$

Question…

- How do you think poverty should change under an association decreasing rearrangement?

Association Decreasing Rearrangement

$$x = \begin{bmatrix} 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{bmatrix} \quad y = \begin{bmatrix} 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{bmatrix}$$

If you think that good health can ***substitute*** (***compensate)*** for bad income or bad education, then poverty should ***decrease***

If you think that good health is ***necessary (complementary)*** to achieve good income and good education, then poverty should ***increase***

If you think that health is not necessary to achieve good income and good education, and can not either substitute for any of these, (i.e., you think they are ***independent***), then poverty should ***not change***.

Axiom Specific to the Multidimensional Case

Weak Rearrangement: If x is obtained from y by an *association decreasing rearrangement among the poor*, $P(y;z) \leq P(x;z)$

What is the assumption behind the axiom?

Assumption: Attributes are ***independent*** (if =) or ***substitutes*** (if <) (compensating achievements)

Could be ***complements***, then axiom should go in the other way (>). (Bourguignon & Chakravarty, 2003).