OPHI
OXFORD POVERTY & HUMAN DEVELOPMENT INITIATIVE
www.ophi.org.uk

UNIVERSITY OF OXFORD

Summer School on Multidimensional Poverty

8–19 July 2013

Institute for International Economic Policy (IIEP)
George Washington University
Washington, DC

Tabita, Kenya · Rabiya, India · Stéphanie, Madagascar · Agathe, Madagascar · Dalma, Kenya · Ann-Sophia, Kenya · Valérie, Madagascar

Properties of Multidimensional Poverty Measures

Suman Seth

Oxford Poverty & Human Development Initiative (OPHI)

Focus of This Lecture

Discuss the properties that are considered 'desirable' for the measurement and understanding of poverty in the multidimensional context

Main Sources of this Lecture

- Bourguignon and Chakravarty (2003): The Measurement of Multidimensional Poverty
- Alkire and Foster (2007, 2011): Counting and Multidimensional Poverty Measurement
- Please see the reading list for others

- Alkire *et al*. (2013): Multidimensional Poverty: Measurement and Analysis, *in progress*

Preliminaries

Multiple dimensions

- Standard of living, knowledge, quality of health (referred as 'achievements')

Achievements of a society or country can be represented by a matrix or joint distribution

Unit of analysis may be individual or household

Preliminaries

A typical dataset or achievement matrix (with 4 dimensions)

		Income	Years of Education	Sanitation (Improved?)	Access to Electricity	
		700	14	Yes	Yes	**Person 1**
x =		300	13	Yes	No	**Person 2**
		400	10	No	No	**Person 3**
		800	11	Yes	Yes	**Person 4**
z =		**500**	**12**	**Yes**	**Yes**	

z is the vector of poverty lines

Preliminaries

Matrix $x=[x_{ij}]_{n\times d}$ summarizes the joint distribution of 'd' attributes across 'n' individuals

Row vector $x_{i\bullet}$ denotes the achievements of person i in all d dimensions

Column vector $x_{\bullet j}$ denotes the achievements in dimension d of all n persons

Vector $z=[z_1,\ldots,z_d]$ be the cut-off vector containing the poverty line of each dimension

Preliminaries

A general achievement matrix

x_{ij}: the achievement of individual i in dimension j

Example:

x_{1d}: the achievement of the 1st individual in dimension d

x_{n1}: the achievement of the nth individual in the first dimension

$$\begin{array}{cc} & \text{Dimensions} \\ x = & \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \cdots & & \\ & & \cdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix} = \begin{bmatrix} x_{1\bullet} \\ x_{2\bullet} \\ \\ \\ x_{n\bullet} \end{bmatrix} \text{ Persons} \\ & \parallel \\ & [\, x_{\bullet 1} \;\; \cdots \;\; x_{\bullet d} \,] \end{array}$$

Measurement

Measurement of multidimensional poverty involves two major steps like unidimensional measurement

- Identification
- Aggregation

First Step: Identification

Identification: *Who is multidimensionally poor*?

An '*identification function*', r, decides who should be multidimensionally poor

$r(x_{i\bullet},z) = 1$ if person i is multidimensionally poor
$r(x_{i\bullet},z) = 0$ if person i is not multidimensionally poor

There can be two types of identification Approaches

Dimension-specific Deprivation Approach (Includes Counting)
Aggregate Poverty Line Approach

First Step: Identification

Identification: Dimension-specific Deprivation Approach

First stage: Determine whether individuals are deprived in each dimension

Second stage: Identify if someone is poor based on an identification function (criterion)

Examples:

Union criterion (if deprived in at least one dimension)

Intersection criterion (if deprived in all dimensions)

Intermediate criterion

First Step: Identification

Example: Constructing first stage 'Deprivation Matrix'

Replace entries: 1 if deprived, 0 if not deprived

	Income	Years of Education	Sanitation (Improved?)	Access to Electricity	
x =	700	14	Yes	Yes	**Person 1**
	300	13	Yes	No	**Person 2**
	400	10	No	No	**Person 3**
	800	11	Yes	Yes	**Person 4**
z =	**500**	**12**	**Yes**	**Yes**	

First Step: Identification

Example: Constructing first stage 'Deprivation Matrix'

Replace entries: 1 if deprived, 0 if not deprived

$$g^0 = \begin{array}{c} \text{Income} & \text{Years of Education} & \text{Sanitation (Improved?)} & \text{Access to Electricity} & \\ \left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right| & \begin{array}{l} \text{Person 1} \\ \text{Person 2} \\ \text{Person 3} \\ \text{Person 4} \end{array} \end{array}$$

$$z = \left|\begin{array}{cccc} \mathbf{500} & \mathbf{12} & \textbf{Yes} & \textbf{Yes} \end{array}\right|$$

<u>These entries fall below cutoffs</u>

First Step: Identification

Example: Equivalently ‘Censored Deprivation Matrix’

$x_{ij}^* = x_{ij}$ if $x_{ij}^* < z_j$ and $x_{ij}^* = z_j$ if $x_{ij}^* \geq z_j$

	Income	Years of Education	Sanitation (Improved?)	Access to Electricity	
$x^* =$	500	12	Yes	Yes	**Person 1**
	300	12	Yes	No	**Person 2**
	400	10	No	No	**Person 3**
	500	11	Yes	Yes	**Person 4**
$z =$	500	12	Yes	Yes	

<u>These entries fall below cutoffs</u>

First Step: Identification

Example:

$$g^0 = \begin{array}{cccc|l} \text{Income} & \text{Years of Education} & \text{Sanitation (Improved?)} & \text{Access to Electricity} & \\ \hline 0 & 0 & 0 & 0 & \text{Person 1} \\ 1 & 0 & 0 & 1 & \text{Person 2} \\ 1 & 1 & 1 & 1 & \text{Person 3} \\ 0 & 1 & 0 & 0 & \text{Person 4} \end{array}$$

$$z = \begin{vmatrix} 500 & 12 & \text{Yes} & \text{Yes} \end{vmatrix}$$

Union? Intersection?

First Step: Identification

Example: Constructing first stage ‘Deprivation Matrix’

Replace entries: 1 if deprived, 0 if not deprived

	Income	Years of Education	Sanitation (Improved?)	Access to Electricity	
$g^0 =$	0	0	0	0	**Person 1**
	1	0	0	1	**Person 2**
	1	1	1	1	**Person 3**
	0	1	0	0	**Person 4**
$z =$	**500**	**12**	**Yes**	**Yes**	

Union? Intersection?

First Step: Identification

Identification: Aggregate Poverty Line Approach

A person is identified as poor if her aggregate achievement falls below an aggregate poverty line

Let the aggregation function be denoted by f

Then,

$$r(x_{i\bullet},z) = 1 \quad \text{if } f(x_{i\bullet}) < f(z)$$
$$r(x_{i\bullet},z) = 0 \quad \text{if } f(x_{i\bullet}) \geq f(z)$$

Example consumer expenditure approach

Note: No deprivation matrix was created in this situation

Second Step: Aggregation

Aggregation: *How poor is the society?*

Based on the identification criterion, this step constructs an index of poverty P(x;z) summarizing the information of the poor (*a censored matrix can be created just as in the unidimensional framework*)

Axioms

Axioms in Multidimensional Context

Two types

1. *Natural extensions* of the unidimensional framework.

2. Axioms specific to the multidimensional context

Natural Extensions

Symmetry *(Anonymity):*

Natural Extensions

Symmetry *(Anonymity):* If matrix y is obtained from matrix x by a *permutation* of achievements and the poverty lines remain unchanged, then P(y;z) = P(x;z)

y is obtained from x by a ***permutation*** of incomes if x = Py, where P is a permutation matrix.

Example: $y = P'x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 4 & 4 & 2 \\ 8 & 6 & 3 \end{bmatrix}$

Natural Extensions

***R**eplication **I**nvariance* *(Population Principle)*:

OPHI Oxford Poverty & Human Development Initiative

UNIVERSITY OF OXFORD

Natural Extensions

***Replication Invariance** (Population Principle)*: If matrix y is obtained from matrix x by a *replication* and the poverty lines remain unchanged, then P(y;z) = P(x;z)

y is obtained from x by a ***replication*** if each person's achievement vector in x is simply repeated a finite number of times

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$ $y = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \\ 8 & 6 & 3 \end{bmatrix}$

Natural Extensions

Scale Invariance *(Homogeneity of Zero-Degree)*:

Natural Extensions

Scale Invariance *(Homogeneity of Zero-Degree)*: If all achievements in matrix x and all poverty lines in z are changed by the same *proportion* a > 0, then P(ax;az) = P(x;z).

Example:

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad z = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix}$$

$$\alpha X = \begin{bmatrix} 2(4) & 2(4) & 2(2) \\ 2(3) & 2(5) & 2(4) \\ 2(8) & 2(6) & 2(3) \end{bmatrix} \qquad \alpha z = \begin{bmatrix} 2(4) & 2(5) & 2(3) \end{bmatrix}$$

Useful but May be controversial!

Natural Extensions

***Focus*:**

Natural Extensions

Focus: Unlike in the unidimensional framework, there are two types of focus axiom

(*Type I*) Focus on those identified as multidimensionally poor' (*we are not interested in those who are not multidimensionally poor*)

(Type II) Focus on dimensions where multidimensionally poor are deprived (*we are not interested in dimensions in which they are not deprived*)

Natural Extensions

Poverty Focus (Type I): If y is obtained from x by *an increment to a non-poor person's achievements* and the poverty lines remain unchanged, then P(y;z) = P(x;z)

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 4 \end{bmatrix}$, $z = (5,6,4)$, and $g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Person 3 is not multidimensionally poor, does it matter if he/she experiences an increase in any of the dimensions?

Natural Extensions

Deprivation Focus (Type II)*:* If y is obtained from x by *an increment in achievements among the non-deprived*, then P(X;z)=P(Y;z). [Recall Deprived vs. Poor]

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 4 \end{bmatrix}$, $z = (5,6,4)$, and $g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Suppose person 2 is considered multidimensionally poor, does it matter if he/she experiences an increment in the third dimension in which he/she is not deprived?

Natural Extensions

Focus Axioms and Types of Identification

Each of the two focus axioms is attributed to a each identification technique introduced earlier

- Poverty focus is attributed to the Aggregated Poverty Line Approach
- Deprivation focus is attributed to the Dimension-specific Deprivation Approach

Natural Extensions

Continuity: For any sequence x^{ν}, if x^{ν} converges to x, then $P(x^{\nu};z)$ converges to $P(x;z)$

A technical assumption. It prevents poverty measures from changing abruptly for changes in distribution of achievements

Similar intuitive interpretation as the assumption in single dimensional framework

Natural Extensions

Monotonicity:

Natural Extensions

Monotonicity: If y is obtained from x by a *deprived increment* among the poor and the poverty line remains unchanged, then $P(y,z) < P(x,z)$

y is obtained from x by a *deprived increment* if there is an increment in a deprived achievement of a multidimensionally poor

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}, z = (5\ \ 6\ \ 4), y = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$

Person 1 is multidimensionally poor, and experiences an improvement in the third dimension.

Natural Extensions

Population Subgroups

Suppose the population size of x is denoted by n(x). Matrix x is divided into two population subgroups: x' with population size n(x') and x'' with population size n(x'') such that n(x) = n(x') + n(x'')

$$
x = \begin{array}{c c}
\begin{array}{ccc} \text{Income} & \text{Education} & \text{Health} \end{array} & \\
\begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} & \begin{array}{l} \text{Person 1} \\ \text{Person 2} \\ \text{Person 3} \end{array}
\end{array}
$$

Natural Extensions

***Population Subgroup Consistency*:**

***Population Subgroup Decomposability*:**

Natural Extensions

Population Subgroup Consistency: If $P(y';z) > P(x';z)$ and $P(y'';z) = P(x'';z)$, and $n(x') = n(y')$, $n(y'') = n(x'')$, then $P(y;z) > P(x;z)$

Population Subgroup Decomposability:

Natural Extensions

Population Subgroup Consistency: If $P(y';z) > P(x';z)$ and $P(y'';z) = P(x'';z)$, and $n(x') = n(y')$, $n(y'') = n(x'')$, then $P(y;z) > P(x;z)$

Population Subgroup Decomposability: A poverty measure is additive decomposable if:

$$P(x) = \frac{n(x')}{n} P(x') + \frac{n(x'')}{n} P(x'')$$

Recall: *decomposability implies subgroup consistency, but the converse does not hold*

Natural Extensions

Transfer in unidimensional context:

Natural Extensions

Transfer in unidimensional context: If y is obtained from x by a progressive transfer among the poor, then $P(y;z) < P(x;z)$

Recall if income is transferred from a person to another who is not richer than the former, keeping mean income same, the transfer is called a *progressive transfer*

This is also known as *Pigou-Dalton* transfer principle

Example: $z = 10$, $x = (9,4,15,8)$; $y = (9,5,15,7)$

Natural Extensions

Transfer in multidimensional context:

Bistochastic matrix (B): A matrix whose row elements and column element sum up to one

Example: A general bistochastic matrix $\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$

Multiply a vector by a bistochastic matrix

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 16 \end{bmatrix} = \begin{bmatrix} 7.6 \\ 8.8 \\ 11.6 \end{bmatrix}$$

Natural Extensions

Transfer in multidimensional context:

Bistochastic matrix (B): A matrix whose row elements and column element sum up to one

Example: What bistochastic matrix is used to obtain y = (9,5,15,7) from x = (9,4,15,8)?

It is $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0.25 \\ 0 & 0 & 1 & 0 \\ 0 & 0.25 & 0 & 0.75 \end{bmatrix}$

Natural Extensions

***Uniform Majorization (UM)*:** Matrix y is obtained from x by a *Uniform Majorization among the poor* (an averaging of achievements among the poor) if y = Bx, where B is an n×n bistochastic matrix but not a permutation matrix, and b_{ii}=1 for every non-poor person *i* in Y.

$$X = BY = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3.5 & 4.5 & 3 \\ 3.5 & 4.5 & 3 \\ 8 & 6 & 3 \end{bmatrix}, \text{ and } z = [5\ \ 6\ \ 5]$$

Achievements of the first two persons (poor) were smoothed

Natural Extensions

Transfer Under UM: If y is obtained from x by a *uniform majorization among the poor* (an averaging of achievements among the poor), then $P(y;z) \leq P(x;z)$.

Axioms Specific to the Multidimensional Context

Axioms Specific to the Multidimensional Context

$$\begin{array}{ccc} & \begin{array}{ccc}\text{Income} & \text{Education} & \text{Health}\end{array} & \\ x = & \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} & \begin{array}{l}\text{Person 1}\\ \text{Person 2}\\ \text{Person 3}\end{array}\end{array}$$

Dimensional Breakdown

It is a *purely multidimensional* concept, where the overall poverty can be expressed as an weighted average of dimensional deprivations of the poor

Axioms Specific to the Multidimensional Context

Dimensional Breakdown

Formally, let $P_j(x_{\cdot j};z)$ summarizes the post-identification (r) deprivation profile of all poor in dimension j

Then, $$P(x;z) = w_1 P_1(x_{\cdot 1};z) + \cdots + w_d P_d(x_{\cdot d};z)$$

where w_j is the weight (normalized) assigned to dimension j

For *union criterion*, it is referred as factor decomposability by Chakravarty, Mukherjee and Ranade (1998)

$$P_j(x_{\cdot j};z) = P_j(x_{\cdot j};z_j)$$

Axioms Specific to the Multidimensional Context

Rearrangements

$$x = \begin{array}{cl} \begin{array}{ccc} \textbf{Income} & \textbf{Education} & \textbf{Health} \end{array} & \\ \begin{bmatrix} 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{bmatrix} & \begin{array}{l} \textbf{Person 1} \\ \textbf{Person 2} \\ \textbf{Person 3} \end{array} \end{array} \qquad y = \begin{array}{cl} \begin{array}{ccc} \textbf{Income} & \textbf{Education} & \textbf{Health} \end{array} & \\ \begin{bmatrix} 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{bmatrix} & \begin{array}{l} \textbf{Person 1} \\ \textbf{Person 2} \\ \textbf{Person 3} \end{array} \end{array}$$

$$z = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix}$$

Is the pattern of poverty same in both societies?

If not, what is the difference?

Axioms Specific to the Multidimensional Context

Both matrices have the same distribution for each dimension (*marginal distribution*)

The correlation between dimensions are not same

Require an axiom based on *correlation/association* between dimension when marginals are same (Atkinson & Bourguignon, 1982; Boland & Proschan, 1988).

This axiom is intrinsic to the multivariate case

Axioms Specific to the Multidimensional Context

$$x = \begin{bmatrix} 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{bmatrix} \quad y = \begin{bmatrix} 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{bmatrix}$$

Ways to call the data transformation:

From x to y:

- association increasing rearrangement
- correlation-increasing transfer
- correlation increasing switch

From y to x:

- association decreasing rearrangement

Axioms Specific to the Multidimensional Context

Question…

How do you think poverty should change under an *association decreasing rearrangement*?

Axioms Specific to the Multidimensional Context

$$x = \begin{bmatrix} 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{bmatrix} \quad y = \begin{bmatrix} 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{bmatrix}$$

If you think that good health can ***substitute*** (***compensate)*** for bad income or bad education, then poverty should ***decrease***

If you think that good health is ***necessary (complementary)*** to achieve good income and good education, then poverty should ***increase***

If you think that health is not necessary to achieve good income and good education, and can not either substitute for any of these, (i.e., you think they are ***independent***), then poverty should ***not change***.

Bourguignon and Chakravarty (2003)

Axioms Specific to the Multidimensional Context

***Decreasing in Association Decreasing Rearrangement*:** If an achievement matrix x' is obtained from another achievement matrix x by an association decreasing rearrangement among the poor, then P(x';z) < P(x;z). [Achievements are assumed to be substitutes]

***Increasing in Association Decreasing Rearrangement*:** If an achievement matrix x' is obtained from another achievement matrix x by an association decreasing rearrangement among the poor, then P(x';z) > P(x;z). [Achievements are assumed to be complements]

There are weak versions as well

Axioms Specific to the Multidimensional Context

Dimensional Monotonicity: If y is obtained from x by a *dimensional increment among the poor*, then P(y,z)<P(x,z)

y is obtained from x by a *dimensional increment among the poor* if due to an increment in a deprived achievement of a poor, he or she becomes non-deprived in that dimension

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}, z = (5\ \ 6\ \ 4), y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 6 & 4 \\ 8 & 6 & 3 \end{bmatrix}$

Suppose person 2 is considered multidimensionally poor, and experiences an increment in the second dimension and is no longer deprived in it

Axioms Specific to the Multidimensional Context

Dimensional Monotonicity:

Why important?

Ordinality vs. cardinality