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Notes on Effective Freedom

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Abstract

This paper presents an intuitive approach for comparing opportunity sets in terms of the extent of the freedom they offer. The decision maker faces a range of scenarios, here modeled as a collection of possible preference orderings over alternatives. One set is said to have greater effective freedom if, for each preference ordering, it contains an alternative that dominates all the alternatives in the second set. The properties of the effective freedom ranking are explored and various full and partial representations are presented. A key example is provided that shows how the effective freedom relation can rank sets even when the preference orderings strongly disagree with one another. Depending on the collection of preferences, the approach can generate the indirect utility ranking, for which unchosen alternatives have no value, or the Pattanaik and Xu (1990) cardinality ranking, in which every alternative has the same intrinsic value.

Keywords: freedom, individual choices, welfare, capabilities, axiomatic approach, orderings.

JEL classification: D63, D03, Z13, I31.

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1. Notation (follows Sen, 1991). Let X be the finite set of alternatives. An opportunity set or menu is any nonempty subset of X, denoted by A, B, C, etc. The set of all menus is $Y = 2^X \setminus \phi$. The main question we are addressing is: When does A have greater freedom than B? In other words, we are in search of a binary ranking R* over Y. There is no presumption that R* must be able to rank any two menus. However, we will typically expect R* to satisfy AR*A for all A∈Y, while AR*C should always follow AR*B and BR*C. Thus, R* will be a quasi-ordering: reflexive and transitive but not necessarily complete.

2. The key to our approach is a set $\mathfrak{R}$ of conceivable (allowable) individual preference orderings on X. Each $R \in \mathfrak{R}$ is assumed to be able to rank any two alternatives in X; each R is a complete ordering. The number of elements of $\mathfrak{R}$ is not specified. $\mathfrak{R}$ may contain only a single ordering R; or $\mathfrak{R}$ could be the set of all logically possible orderings over X.

An individual with ordering R can judge a menu according to its best element or elements. If so, then R can be extended from X to Y as follows

$$ARB \text{ iff } \max_R A \; R \; \max_R B,$$

where $\max_R A$ is any best element of A. This is what Bossert, Pattanaik and Xu call the "indirect utility" approach to extending an ordering. Our notion of effective freedom is based on the following definition.

Definition. One menu A has as much effective freedom as another menu B, written AR*B, whenever each of the allowable preference orderings in $\mathfrak{R}$ considers A to be as preferable as B. In other words,

$$AR^*B \text{ iff } ARB \text{ for all } R \in \mathfrak{R}.$$

Clearly AI*B whenever all preference orderings consider A to be indifferent to B. Also, AP*B if all find A to be as good as B, and at least one considers A to be strictly preferred to B. The menus A and B cannot be compared using R* if at least one ordering in $\mathfrak{R}$ (strictly) prefers A to B and another (strictly) prefers B to A. In particular, over singleton sets R* is the standard unanimity ranking so that {x}R*{y} holds when all agree that xRy; and {x} is incomparable to {y} if at least two orderings (strictly) disagree.

3. Properties. R* is an example of what might be called a unanimity (or intersection) ranking. As such, we know that it is a quasi-ordering and that it can be represented by a vector-valued function. Other properties include:

A. $A \supseteq B$ implies AR*B

B. [AR*B and CR*D] implies $(A \cup C)R^*(B \cup D)$.

C. AR*B implies $(A \cup B)I^*A$

D. If A and B are noncomparable under R*, then $(A \cup B)P^*A$.

Proofs

A. If $A \supseteq B$, then $\max_R A \; R \; \max_R B$ for each $R \in \mathfrak{R}$ and so AR*B.

B. Let a be a best element of A, let c be a best element of C, and let x be a best element of $A \cup C$ according to R. Pick any $y \in B \cup D$. If $y \in B$ then xRa and aRy so that xRy by transitivity.

If $y \in D$ then xRc and cRy so that xRy again by transitivity. Hence for any given R we have $\max_R(A \cup C)$ R $\max_R(B \cup D)$, and so $(A \cup C) R^* (B \cup D)$.

C. Applying property B to AR^*B and AR^*A yields $AR^*(A \cup B)$. Property A implies $(A \cup B) R^*A$.

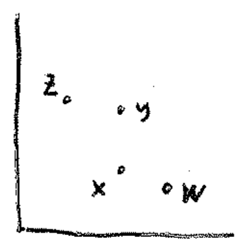
D. Property A yields $(A \cup B) R^*B$, hence $(A \cup B) RB$ for all $R \in \mathfrak{R}$. By the noncomparability of A and B there is some $R \in \mathfrak{R}$ such that BPA. By transitivity, $(A \cup B)$ PA, and hence $(A \cup B) P^*A$ by the definition of P^*.

Property A is called "weak set dominance" by Sen (1991). It requires that more choices lead to a higher (or the same) level of freedom. Property B might be called "composition" after Sen (1991). It says that ranking R* is preserved under union. Property C points out the irrelevance of adding dominated alternatives. It is motivated by Kreps, p. 568. In words, if one menu has more (or the same level of) effective freedom than another, then it (the former menu) has the same level of freedom as the union of the two. (Actually, the converse is also true: if adding a set of elements results in an unchanged level of freedom, then the original menu must have had more (or the same level of) effective freedom than the collection of added elements). Property D points out the advantage of adding noncomparable menus. If A and B are unranked by R*, then combining the two menus results in a new opportunity set with strictly more freedom than either A or B alone. When the preference orderings cannot agree whether A or B is more desirable, they unambiguously rank $A \cup B$ above either of the two; but when they all agree that A is as good as B, then adding B to A yields no improvement in effective freedom.

4. Completeness of R*. In general, the quasi-ordering R* is incomplete, with the amount of incomparability depending on the extent to which preferences in $\mathfrak{R}$ agree. For example, if $\mathfrak{R}$ contains only a single ordering R, then R* is simply the extension of R to Y and hence it is complete. At the other extreme, where $\mathfrak{R}$ contains all logically possible orderings on X, the only menus that can be compared are ones related by set inclusion. For if A contains an alternative x not in B, and B contains an alternative y not in A, then at least one ordering in $\mathfrak{R}$ strictly prefers x to every point in B while another ordering strictly prefers y to A. Hence A and B are noncomparable according to R*, and so: $A \supseteq B$ iff AR*B. This example exhibits the largest degree of incompleteness allowed by Property A.

An intermediate example shows the potential utility of the effective freedom approach. Let C = {g,t,w} and D = {b,a,d} where every alternative in C is regarded by each $R \in \mathfrak{R}$ as being strictly preferred to every alternative in D. The relative rankings of g, t, w are assumed to vary across the orderings in $\mathfrak{R}$, as do the rankings of b, a, d. It may help to think of the alternatives in C as "great," "terrific," and "wonderful" and the alternatives in D as "bad," "awful," and "dismal," as in Sen (1990). The resulting R* goes beyond set inclusion in the following ways: (1) If A has an alternative from C, and B doesn't, then AR*B. (2) If both have alternatives from C and $A \cap C \supseteq B \cap C$, then AR*B. In particular, the menus C and D themselves are comparable, and are ranked CP*D as one would expect.

One final example will show that R* can make useful comparisons even when preferences in $\mathfrak{R}$ utterly disagree over X. Let X be a finite set in the cartesian plane and let $\mathfrak{R}$ contain two orderings: R_1 represented by $u_1(x) = x_1 - x_2$; and R_2 represented by $u_2(x) = x_2 - x_1$. Clearly R_1 regards x to be preferred to y exactly when y is preferred to x according to R_2. So if X = {w,x,y,z} as depicted below, the two orderings on X never agree: R_1 ranks the elements of X as $wP_1xP_1yP_1z$ while R_2 ranks the alternatives $zP_2yP_2xP_2w$. The unanimity ranking over elements of X is empty. However, the unanimity ranking R* over <u>subsets</u> of X goes beyond simple set inclusion. For instance, {w,z} P* {x,y}, since according to R_1 both x and y are strictly dominated by w, while z is preferred to x and y under R_2. Moreover, since singleton sets are noncomparable, any two-alternative menu has greater effective freedom than either singleton submenu.



5. <u>Representation of R*</u>. We have seen how adding alternatives to a given menu may well leave effective freedom unchanged; in other words, a set A and a larger set B may satisfy AI*B. Indeed, a menu that is indifferent to A under R* may be a superset of A, a subset of A, or neither. However, there is one special indifferent menu that will play an important role in describing R*. Let $D(A) = \cup_{AR^*B} B$. D(A) might be called the "free disposal" set of A since it includes all points from sets weakly dominated by A under R*. This set has the following useful properties.

Result For any $A,B\in Y$,

(i) $AI^*D(A)$

(ii) AR^*B iff $D(A) \supseteq D(B)$.

Proof

(i) Note that $D(A) \supseteq A$ so that $D(A)\ R^*A$ by Property A. Conversely, consider any two sets B and B′ for which AR^*B and AR^*B'. Property B yields $AR^*(B \cup B')$. Noting that D(A) is just the finite union of sets B for which AR^*B, we obtain $AR^*D(A)$ by repeated application of Property B. Thus $AI^*D(A)$.

(ii) Suppose that AR^*B. By transitivity of R*, any C for which BR^*C must also satisfy AR^*C. Hence $\cup_{AR^*C} C \supseteq \cup_{BR^*C}$ which gives us $D(A) \supseteq D(B)$. Conversely, suppose that $D(A) \supseteq D(B)$. Then by Property A we have $D(A)R^*D(B)$, and thus by (i) and transitivity AR^*B.

Result (i) implies that D(A) is the largest set indifferent to A under R*. When choosing from A, it is "as if" one were choosing from D(A). Consequently, D(A) might also be called the <u>effective</u> opportunity set associated with A. Result (ii) indicates the crucial role effective opportunity sets play in determining when R* applies. To check whether A is comparable with B, one simply checks whether D(A) and D(B) are related under set inclusion. If not, then A and B cannot be compared. If so, then AR^*B or BR^*A accordingly as $D(A) \supseteq D(B)$ or $D(A) \subseteq D(B)$. Thus, in an intuitive sense, $D(\cdot)$ may be said to "represent" the quasi-ordering R*.

6. <u>Vector Representation of R*</u>. Economists find it more natural to represent orderings by "utility functions." Since R* is only a quasi-

ordering, either the definition of representation or the comparison space must be altered to accommodate potential incompleteness. We choose the latter. A vector valued function u: $Y \rightarrow \mathbf{R}^n$ is said to vector-represent R* if

$$AR^*B \text{ iff } u(A) \geq u(B),$$

where "$\geq$" is vector dominance in $\mathbf{R}^n$. It turns out that R* has a very simple vector-representation. Let n be the number of elements in X. Enumerate the elements of X from 1 to n. Define u: $Y \rightarrow \mathbf{R}^n$ by $u_i(A) = 0$ if the ith element of X is not in D(A), and $u_i(A) = 1$ if the ith element is in D(A). The following result is immediate.

Result. The function $u(\cdot)$ vector-represents the quasi-ordering R*.

Proof. Clearly $u(A) \geq u(B)$ iff $D(A) \supseteq D(B)$. From Result (ii), then, $u(A) \geq u(B)$ iff AR*B.

Note. The dimension of u is n, which is significantly smaller than 2^n, the dimension of Y. In some cases one may be able to find a vector-representation with even fewer dimensions. For example, if $\mathfrak{R}$ has a single element, then R* is complete and therefore one can find a real-valued function to represent R*. A common way to do this is to first define f(x) as the number of elements in the lower contour set of x, and then extend f to Y by letting f(A) be the number of elements in the lower contour set of a best element of A. Interestingly, f(A) is then just #D(A), the number of elements in the effective opportunity set D(A), or $f(A) = \Sigma_i u_i(A)$. We will return to this "counting representation" below.

7. Conjecture. If R^o is a quasi-ordering satisfying Properties A and B, then there exists a collection $\mathfrak{R}$ of complete orderings such that R^o is the R* associated with $\mathfrak{R}$.

Note. Perhaps this will be proved along the lines of Kreps who assumed a complete ordering.

8. Counting Representation. If R* is incomplete, the counting representation f(A), which assigns A the number #D(A), is not a full-blown representation of R*. In fact, f only partially represents R* in the sense that

$$AI^*B \text{ implies } f(A) = f(B)$$

and

$$AP^*B \text{ implies } f(A) > f(B).$$

The converse implications do not hold.

The counting representation induces a complete ordering, say $R^\#$ on Y, which extends R* in a natural way. If two menus A and B cannot be directly ranked via R*, simply count up the number of alternatives in the effective opportunity sets D(A) and D(B). The set with the greater number of elements in its effective opportunity set is ranked higher according to $R^\#$. As noted before, if R* is complete then $R^* = R^\#$. If R* is incomplete, than $R^\#$ is one possible way of extending R* to a complete ordering.

9. An Interpretation of Pattanaik and Xu. The criterion obtained by Pattanaik and Xu (1990) ranks sets according to the number of alternatives they contain. The underlying idea appears to be that more choice, regardless of the quality of the additional alternatives, means

greater freedom. This would seem to fundamentally contradict the "effective freedom" approach which carefully screens out ineffective alternatives before making judgments. Yet our framework also admits the ordering obtained by Pattanaik and Xu (1990), so long as we make the following assumptions: (1) $\mathfrak{R}$ is the set of all logically possible orderings on X, and (2) R* is extended to $R^{\#}$ using the counting representation. In the special case where $\mathfrak{R}$ is unrestricted, the resulting ordering R* is set inclusion and the effective opportunity set of A is A itself. If, on top of this assumption, we judge a menu by the number of elements in its effective opportunity set, we obtain the Pattanaik and Xu ordering $R^{\#}$.

The counting representation is an arbitrary, but not terribly objectionable, method of extending R* to a complete ordering. However, the assumption of perfect ignorance about preferences is problematic and, indeed, is lurking behind the many criticisms of Pattanaik and Xu's result.

References

Bossert, W., Pattanaik, P.K., and Xu, Y., Ranking opportunity sets: an axiomatic approach, Mimeo, May 1992.

Kreps, D. M., A representation theorem for "preference for flexibility," Econometrica, Vol. 47, No. 3, 565-577, May 1979.

Pattanaik, P.K., and Xu, Y., Informational basis for evaluation of freedom of choice, Mimeo, April 1992

Pattanaik, P.K., and Xu, Y., On ranking opportunity sets in terms of freedom of choice, Recherches Economiques de Louvain, Vol. 56, No. 3-4, 383-90, 1990.

Sen, A., Inequality Reexamined, Clarendon Press, Oxford, 1992.

Sen, A., Welfare, Freedom and Social Choice: a reply, Recherches Economiques de Louvain, Vol. 56, No. 3-4, 451-485, 1990.

Sen, A., Markets and freedoms, Harvard Discussion Paper No. 1559, July 1991.

Sen, A.K., Welfare, preference and freedom, Journal of econometrics, Vol 50, 15-29, Oct/Nov 1991.

Sen, A.K., Freedom of choice: concept and content, European Economic Review, Vol 32, 269-94, March 1988.