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Multidimensional Inequality and Human Development

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Abstract

The measurement of inequality from a human development perspective is fundamental. We start this paper by briefly introducing the human development approach and its main conceptual basis: the capability approach. We note that inequality should preferably be assessed in the space of functionings, requiring the assessment methods to use multidimensional techniques. We then present the primary challenges inherent to multidimensional inequality measurement that are related to two types of distributional changes: one is concerned with the dispersions within distributions that are analogous to the unidimensional framework and the other, unlike the unidimensional framework, is concerned with the association between distributions. We next present a succinct review of the most prominent measures proposed in the literature within a unifying framework and review the empirical applications surrounding these measures. We note that while multidimensional inequality measures have a great potential to contribute to the monitoring of human development, there are some challenges to overcome in order to fulfil this potential.

Keywords: multidimensional inequality measurement, capability approach, transfer principles, multidimensional association, normative framework, human development.

JEL classification: O15, D63, I32.

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The Oxford Poverty and Human Development Initiative (OPHI) is a research centre within the Oxford Department of International Development, Queen Elizabeth House, at the University of Oxford. Led by Sabina Alkire, OPHI aspires to build and advance a more systematic methodological and economic framework for reducing multidimensional poverty, grounded in people's experiences and values.

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1. Measurement of Inequality within a Human Development Framework

The human development approach became widely known and prevalent in the international development agenda with the first Human Development Report (HDR), published by the United Nations Development Programme (UNDP) in 1990. In fact, the Human Development Index (HDI), which is published annually in such reports, has become an important reference metric. The essence of the human development approach is that development must have human beings at its centre. Therefore, it is imperative to capture the distribution of human development across people, in addition to evaluating the level of human development as is done by indices such as the HDI. In other words, there is a need to measure inequality, and this must be done in the same space where human development is measured. We hope this paper will be a useful introduction to the field of inequality measurement within a human development perspective, highlighting the challenges as well as some of its most controversial issues and offering a review of the most prominent measures proposed thus far.

The human development approach has been motivated by and greatly benefited from many conceptual frameworks that go back as far as Aristotle and include the basic needs approach, the Social Doctrine of the Catholic Church, human rights and sustainable livelihoods, just to mention a few. Yet it has been fundamentally shaped and strengthened by Amartya Sen's capability approach (Alkire and Deneulin, 2009), in which development is defined as 'the process of expanding the real freedoms people enjoy' (Sen, 1999, p.3).[1] This conceptualisation has strong implications because it expands the space for evaluating development beyond mere ownership of resources.

Evaluating development within the space of resources, be it income or Rawlsian primary goods, is problematic because these are mere means to ends, not ends in themselves. Moreover, people have different abilities to convert a given amount of income or another specific resource into a certain achievement.[2] Evaluating development within the space of utilities is also problematic because there are **adaptive preferences** whereby people in an objective state of deprivation can show high utility levels. Thus, Sen argues that the space for evaluating development must be that of capabilities and functionings and therefore it is inherently multidimensional.

Functionings are 'the various things a person may value ***doing*** or ***being***', which range from fundamental ones such as being adequately nourished and being free from preventable diseases, to more complex ones

[1] It is worth noting that human development in the HDRs has traditionally been defined as the process of enlarging people's choices. Alkire and Deneulin (2009, p. 34) have clarified that these terms are used for simplicity but do not imply that development is about expanding all or any choice, nor of expanding individual choices necessarily. Development is about expanding the range of valuable possible beings and doings, i.e., expanding the quality of human life.

[2] Such abilities are influenced by personal heterogeneities, environmental diversities, variations in social climate, differences in relational perspectives and intra-household distribution (Sen, 1997).

such as taking part in the life of the community and having self-respect. As long as a person's functionings can be expressed by real numbers, the functionings can be summarised by a **functioning vector**. The set of all functioning vectors available to the person form the person's **capability set** or **capabilities**. One particular functioning vector, or a combination of functionings that the person actually chooses from the set of capabilities, reflects that person's **achievements**. Similar to the concept of a budget set in consumer theory that represents all possible commodity bundles that are affordable to a person, the capability set is the collection of all available functionings or the **set of opportunities** and thus represents the person's freedom to choose or achieve various functionings (see Sen, 1997, pp. 394–95).

Sen favours using the 'capability set' over the chosen or 'achieved functionings' as the space for evaluating development because achieved functionings are merely an element of the entire capability set. The capability set, in contrast, contains *all* available functioning vectors, even those not chosen. This distinction is relevant because two persons may have been observed to choose the same functioning vector and yet one may have chosen the functioning vector in the absence of any better available alternative (i.e. lacks freedom to choose), whereas the other may have chosen the functioning vector despite having better available alternatives (i.e. has freedom to choose).[3] Thus, using the capability set over achieved functioning captures a person's freedom to choose from various alternative functionings regarded as intrinsically valuable (Sen, 1985, 1995).[4]

While Sen favours looking 'at opportunities in an adequately broad way' (Sen, 2002), that is, the capability set, Fleurbaey (2004) votes for refined functionings as defined by Sen (1985, 2002). ***Refined functionings*** contain both achieved functionings and the capability set (Sen, 2002; Fleurbaey, 2004, p.6).[5] Moreover, Fleurbaey (2004) argues that evaluating development using the capability set or the set of available opportunities without considering achieved functionings fits into an ***equality of opportunity*** framework as proposed by Romer (1998). In essence, the equality of opportunity framework distinguishes between inequality caused by factors beyond personal control and inequality due to personal decisions and efforts. However, as people are held accountable for their own wellbeing, the equality of opportunity framework

3 A frequent example offered by Sen is that of two persons with low nutritional status, one is due to the lack of resources and another because of the decision to fast.

4 For Sen, functionings are things people value (the individual values certain functionings) *and* 'have reason to value', implying that social choices need to be made regarding beings and doings that can be considered valuable. For further discussion, see Alkire and Deneulin (2009, ch. 2).

5 Pattanaik and Xu (2007) also use this achievement-opportunity combination for their analysis of the incompatibility between minimal relativism and weak dominance.

may entail a potential contradiction between personal responsibility and giving priority to those who are the worst off (Fleurbaey, 2004).[6]

Interestingly, Fleurbaey contends that the space of functionings allows measuring freedoms and also capturing personal responsibility by putting different weights on functionings. One may evaluate freedoms, he suggests, by putting substantially higher relative weight on basic functionings for human flourishing and by combining this with knowledge of the place where the individual lives. For example, poor educational achievements, low income, and unsatisfactory social relations inevitably reflect the lack of freedom to choose.[7] In fact, this argument of Fleurbaey agrees with that of Sen (1985). In turn, by placing zero or low weights on non-basic functionings, one may give personal responsibility a role.[8]

In this way, we are soon into the practical challenges faced when shaping a measure – be it a measure of human development, inequality or poverty. One first practical challenge relates to a long-standing discussion on whether there should be a list of 'central capabilities' and thus of implied functionings (as required by Fleurbaey). Some capabilities, according to Sen, may be considered basic, such as the ability to move, to meet one's nutritional requirements, to be clothed and sheltered, and to participate in the social life of the community (Sen, 1979, p. 218). Yet, Sen claims that no particular list should be prescribed, mainly because any list needs to be defined according to the purpose of the evaluation. Furthermore, such a list must emerge from deliberative engagement and must necessarily be contingent on time and space (Sen, 2004).[9] In contrast, Martha Nussbaum argues that a list of central human capabilities is fundamental to avoiding issues of omission and power in which people may learn not to want or value certain functionings (Alkire and Deneulin, 2009).[10] At this point it is worth noting that capabilities refer to different **dimensions** of wellbeing (also sometimes called domains), and within them, there may be one or more **indicators** that proxy the capabilities. Choosing dimensions and indicators to be considered in a measure is a key step. Interestingly, Alkire (2008) points out that, in practice, one finds a striking degree of commonality between different lists of central human capabilities or dimensions that have been suggested.[11]

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6 According to Fleurbaey (2004), '...if the concern for personal responsibility must force social institutions to make a distinction between deserving and undeserving poor, some of the worst-off are likely to be left behind, possibly on the wrong grounds' (p. 1). See Ferreira and Peragine (2015) for a review of the theory and evidence of the equality of opportunity approach.

7 As Fleurbaey argues, the difference in the degree of freedom between the person who starves and the one who fasts can be deduced from their achievements in other dimensions, aside from their nutritional status.

8 Additionally, Fleurbaey elaborates and justifies that this does not necessarily imply a paternalistic approach (referred to in his paper as 'perfectionism').

9 For further discussion on this matter, see Alkire et al. (2015, chapter 6).

10 Nussbaum (2000) considers the following list of ten central capabilities: (1) life, (2) bodily health, (3) bodily integrity, (4) sense, imagination, and thought, (5) emotions, (6) practical reason, (7) affiliation, (8) other species, (9) play, and (10) control over one's environment.

11 See Alkire et al. (2015, ch. 6) for further discussion on this matter.

A second practical challenge, closely related to the first one, is the selection of relative weights. Non-included dimensions receive a zero weight. In turn, weighting the included dimensions and indicators also has important implications as it determines their trade-offs. Decancq and Lugo (2012a) review and classify the approaches used to setting weights. Sen (2009) acknowledges the difficulty in setting weights and advises using ranges of weights on which there is some agreement, even if this is far from total. Moreover, a good practice is to perform robustness analysis on (reasonable) changes in the weighting structure.[12] However, explicit weights are not the only determinant of trade-offs across dimensions, so are normalization procedures and the aggregation function across dimensions (Decancq and Lugo, 2012a). All these are non-trivial normative decisions in the evaluation of multidimensional wellbeing, inequality, or poverty that require sound justification and transparency.[13] Analysis and discussion of these matters can be found elsewhere.[14]

In what remains of this paper, we focus on discussing different kinds of methodological and normative issues in multidimensional inequality measurement. After setting the framework of unidimensional inequality measurement in Section 2, we move to the multidimensional inequality measurement framework and present related challenges in Section 3. In Section 4, we discuss various axiomatic properties and provide a succinct review of the most prominent measures in a unifying framework. In Section 5, we provide a review of empirical applications of multidimensional inequality indices. Finally, Section 6 concludes.[15]

2. Inequality within Unidimensional Framework

Initially, let us suppose that human development can be assessed by only a single dimension, which may be either earned incomes or educational attainments. Suppose, there are n (≥ 2) persons in a hypothetical society. For simplicity of presentation, we assume that each of the n persons has an achievement. We denote the achievement of person i by $x_i \in \mathbb{R}_{++}$ for all $i = 1, \ldots, n$, where $\mathbb{R}_{++}$ is the set of strictly positive real numbers. We thus assume that achievements are strictly positive.[16] The collection of all n persons' achievements in the society can be represented by an achievement vector $\mathbf{x} = (x_1, \ldots, x_n) \in$

12 For further discussion on this matter, see Alkire et al. (2015, ch. 6) and Seth and Villar (2017a).

13 In relation to this, one may wonder about the difference between measures of multidimensional inequality and measures of multidimensional poverty. The difference can by synthesized into one word: thresholds. Poverty measurement necessarily entails a dichotomization of the population into poor and non-poor according to some agreed standard of minimum level of satisfaction (Sen, 1976; Foster et al. 2011). On the contrary, the measurement of inequality requires considering the full distribution of achievements.

14 Alkire (2008); Alkire et al. (2015, ch.6); Decancq and Lugo (2012a); Santos and Santos (2014).

15 It is worth acknowledging the related work by Chakravarty and Lugo (2016) and more theoretical discussions by Zoli (2009).

16 When implementing inequality measures empirically, one may encounter negative or zero income values, which require special treatment for certain inequality measures.

$\mathbb{R}^n_{++}$. An achievement vector may be referred to as a **distribution of achievements**. A higher value of achievement reflects a higher level of wellbeing. We denote the average of all achievements in distribution $\mathbf{x}$ by $\mu(\mathbf{x}) = (x_1 + \cdots + x_n)/n$.

Inequality in any single dimension is mainly understood through either a Pigou-Dalton progressive transfer or regressive transfer. A Pigou-Dalton progressive transfer takes place whenever one distribution is obtained from another distribution through a rank preserving transfer of achievement from a person with higher achievement to a person with lower achievement, while keeping the mean achievement unchanged. Consider two distributions: $\mathbf{x} = (1,2,8,9)$ and $\mathbf{y} = (2,2,8,8)$. Note that $\mu(\mathbf{x}) = \mu(\mathbf{y}) = 5$. The primary difference between the two distributions is that $\mathbf{y}$ can be obtained from $\mathbf{x}$ by transferring one unit of achievement from the person with nine units to the person with one unit. In this case, $\mathbf{y}$ is stated to be obtained from $\mathbf{x}$ by a progressive transfer. Technically, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}$, $\mathbf{y}$ is stated to be obtained from $\mathbf{x}$ by a Pigou-Dalton progressive transfer if there are two persons i_1 and i_2 such that $x_{i_1} > x_{i_2}$, $y_{i_1} = x_{i_1} - \delta$ and $y_{i_2} = x_{i_2} + \delta$ for any $\delta > 0$ yet $y_{i_1} > y_{i_2}$, and $y_i = x_i$ for all $i \neq i_1, i_2$.

A Pigou-Dalton regressive transfer, on the other hand, is said to have taken place whenever one distribution is obtained from another distribution through a transfer of achievement from a person with lower achievement to a person with higher achievement, while keeping the mean achievement unchanged. Distribution $\mathbf{x}$, in the example, can be seen as being obtained from distribution $\mathbf{y}$ by a regressive transfer of one unit of achievement from a poorer person to a richer person. Technically, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}$, $\mathbf{x}$ is stated to be obtained from $\mathbf{y}$ by a Pigou-Dalton regressive transfer if there are two persons i_1 and i_2 such that $y_{i_1} > y_{i_2}$, $x_{i_1} = y_{i_1} + \delta$ and $x_{i_2} = y_{i_2} - \delta$ for any $0 < \delta < y_{i_2}$, and $x_i = y_i$ for all $i \neq i_1, i_2$.

Whenever a distribution is obtained from another distribution by a sequence of Pigou-Dalton progressive (regressive) transfers, then inequality in the former distribution is lower (higher) than that in the latter distribution.

Pigou-Dalton progressive transfer(s) can be technically expressed using T-transformation(s). A **T-transformation matrix** (**T**) is a weighted average of an identify matrix **E** and a non-identity permutation matrix **P**, such that $\mathbf{T} = \lambda\mathbf{E} + (1 - \lambda)\mathbf{P}$ where $0 < \lambda < 1$.[17] A permutation matrix is a non-negative square matrix with each row and each column having exactly one element equal to one and the rest being

[17] We define T-transformation here in a strict sense by restricting λ to lie between 0 and 1. Whenever, $\lambda = 1$, a T-transformation matrix coincides with an identity matrix, resulting in no change in the distribution. Whenever, $\lambda = 0$, a T-transformation matrix coincides with a permutation matrix, where elements within an achievement vector merely swap places.

equal to zero. The following combination of $\mathbf{E}$, $\mathbf{P}$, and λ provides the T-transformation matrix for obtaining $\mathbf{y} = (2,2,8,8)$ from $\mathbf{x} = (1,2,8,9)$:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \lambda = 0.875.$$

Thus,

$$\mathbf{T} = \begin{bmatrix} 0.875 & 0 & 0 & 0.125 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.125 & 0 & 0 & 0.875 \end{bmatrix} \text{ and so } \mathbf{y} = \mathbf{xT}.$$

Distribution $\mathbf{y}$ in this case is obtained from distribution $\mathbf{x}$ by post-multiplying $\mathbf{x}$ by a T-transformation matrix. Whenever a distribution is obtained from another distribution by a sequence of Pigou-Dalton transfers, then the former distribution can be equivalently obtained from the latter by a finite number of T-transformations.

The lowest level of inequality or the situation of perfect equality is accomplished whenever everybody has the same level of achievement. Technically, the situation of perfect equality in $\mathbf{x}$ is reached whenever every person has an achievement equal to $\mu(\mathbf{x})$; we denote the equally distributed distribution corresponding to $\mathbf{x}$ as $\bar{\mathbf{x}}$, where $\bar{x}_i = \mu(\mathbf{x})$ for all $i = 1, \ldots, n$. A sequence of T-transformations may lead to the situation of perfect equality. For example, a sequential application of the following two T-transformation matrices leads to $\bar{\mathbf{x}} = (5,5,5,5)$ from $\mathbf{x} = (1,2,8,9)$, i.e., $\bar{\mathbf{x}} = \mathbf{x}\mathbf{T}^1\mathbf{T}^2$:

$$\mathbf{T}^1 = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix} \text{ and } \mathbf{T}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ [18]

We now introduce the related concept of a **bistochastic matrix** (denoted by $\mathbf{B}$) that we will use in subsequent sections. A bistochastic matrix is a non-negative square matrix where each row and each column sums to one. T-transformation matrices themselves as well as a product of T-transformation matrices are bistochastic matrices. Permutation matrices (which include identity matrices) are also bistochastic matrices. Post-multiplying a distribution by a non-permutation bistochastic matrix does not change the mean of the distribution but makes the distribution more equal. However, it should be borne in mind that not all bistochastic matrices, especially those with $n \geq 3$ dimensions, can be expressed as a

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[18] These two T-transformation matrices are not the unique set of matrices for obtaining $\bar{\mathbf{x}}$ from $\mathbf{x}$.

product of T-transformation matrices. An example of a bistochastic matrix that is neither a T-transformation matrix nor a product of T-transformation matrices is

$$\mathbf{B} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$[19]

This difference is not crucially important in the unidimensional context but becomes important in the multidimensional context.

Can the understanding of inequality within the unidimensional context be extended straightforwardly to understanding inequality within multiple dimensions? Does an increase or decrease in inequality within each of the many dimensions lead to an increase or decrease in overall inequality? We critically examine these questions in the next section.

3. Inequality Involving Multiple Dimensions

We introduce some additional notation that is specific to the multidimensional framework. Suppose, in addition to n (≥ 2) persons in the society, inequality is assessed by d (≥ 2) dimensions. Similar to the unidimensional framework, we denote the achievement of person i in dimension j by $x_{ij} \in \mathbb{R}_{++}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, d$, where a higher value of x_{ij} denotes higher achievement within dimension j. The collection of all persons' achievements in a society can be represented by an $n \times d$-dimensional achievement matrix $\mathbf{X}$ as

$$\begin{array}{cc} & \textbf{Dimensions} \\ \mathbf{X} = & \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix} \textbf{People} \end{array}$$

We denote each row i of $\mathbf{X}$ by a d-dimensional vector $\mathbf{x}_{i\cdot}$ summarising person i's achievements in all d dimensions; whereas we denote each column j of $\mathbf{X}$ by an n-dimensional vector $\mathbf{x}_{\cdot j}$ summarising the achievements for all n persons in dimension j. A column vector of achievements is referred to as a **marginal distribution of achievements**, and an achievement matrix, which contains all marginal distributions, is referred to as a **joint distribution of achievements**. For definitional purposes, we will denote the set of all possible matrices of size $n \times d$ by $\mathcal{X}_n \in \mathbb{R}_+^{n \times d}$ and all possible achievement matrices by $\mathcal{X} = \bigcup_n \mathcal{X}_n$. Like in the unidimensional framework, we let $\mu_j(\mathbf{X}) = \mu(\mathbf{x}_{\cdot j})$ denote the average of all achievements in dimension j. The average achievements across all d dimensions are summarised by vector

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[19] See Marshall and Olkin (1979), p. 23.

$\boldsymbol{\mu}(\mathbf{X}) = \big(\mu_1(\mathbf{X}), \ldots, \mu_d(\mathbf{X})\big)$. We also define the additional vector notation. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^d$, $\mathbf{a} \geq \mathbf{b}$ implies that $a_j \geq b_j$ for all j and $\mathbf{a} > \mathbf{b}$ implies that $a_j \geq b_j$ for all j and $a_j > b_j$ for some j.

Let us now see if the concept of a Pigou-Dalton progressive transfer in the unidimensional framework can be extended to the multidimensional framework. We have already discussed in Section 3 that Pigou-Dalton progressive transfers can be presented using T-transformations. In the multidimensional context, similarly, Pigou-Dalton progressive transfer(s) may take place uniformly across all dimensions. For two joint distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a **uniform Pigou-Dalton progressive transfer** (UPDT) whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by pre-multiplying $\mathbf{X}$ by a T-transformation matrix $\mathbf{T}$, i.e., $\mathbf{Y} = \mathbf{TX}$.[20] The **UPDT majorization** requires inequality to be lower if a distribution is obtained from another distribution by a UPDT or a sequence of UPDTs. In the following example, $\mathbf{Y}^1$ is obtained from $\mathbf{X}^1$ by pre-multiplying $\mathbf{X}^1$ by $\mathbf{T}$.

$$\mathbf{Y}^1 = \begin{bmatrix} 2 & 3 \\ 2 & 3 \\ 8 & 9 \\ 8 & 9 \end{bmatrix}; \mathbf{X}^1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 8 & 9 \\ 9 & 10 \end{bmatrix} \text{and } \mathbf{T} = \begin{bmatrix} 0.875 & 0 & 0 & 0.125 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.125 & 0 & 0 & 0.875 \end{bmatrix}.$$

Note that the same T-transformation has been applied uniformly to both marginal distributions; i.e., $\mathbf{y}_{\cdot j}^1 = \mathbf{T}\mathbf{x}_{\cdot j}^1$ for $j = 1,2$. Clearly, $\mu_j(\mathbf{Y}^1) = \mu_j(\mathbf{X}^1)$ for $j = 1,2$.

We have already discussed that the T-transformations and the product of T-transformations can be seen as bistochastic transformations, but not all bistochastic matrices can be presented as products of T-transformation matrices. This is not a problem in the unidimensional context, but it is crucial in the multidimensional context since transformations of some achievements matrices into others can never be obtained by UPD. A concept referred to as **uniform majorization** has thus been introduced in the literature (Kolm 1977). For any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is stated to be obtained from $\mathbf{X}$ by **uniform bistochastic transformation** (UBT) whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by pre-multiplying $\mathbf{X}$ by a bistochastic matrix $\mathbf{B}$; i.e., $\mathbf{Y} = \mathbf{BX}$. Then uniform majorization requires inequality to be lower if a distribution is obtained from another distribution by a UBT. As all T-transformation matrices are bistochastic matrices, then $\mathbf{Y}^1$ can be obtained from $\mathbf{X}^1$ by UM. Again, note that the same transformation has been applied uniformly to both dimensions, i.e., $\mathbf{y}_{\cdot j}^1 = \mathbf{B}\mathbf{x}_{\cdot j}^1$ and also $\mu_j(\mathbf{Y}^1) = \mu_j(\mathbf{X}^1)$ for $j = 1,2$.

In the previous example, each marginal distribution in $\mathbf{Y}^1$ has become more equal than the respective distribution in $\mathbf{X}^1$. Within each marginal distribution, the poorest person is better off at the cost of the

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20 In the unidimensional context, we presented the distribution across persons as a row vector, but in the multidimensional context each marginal distribution across persons is a column vector.

richest person being worse off, while leaving the mean achievement unchanged. Should we consider distribution $\mathbf{Y}^1$ to be more equal than distribution $\mathbf{X}^1$? The answer should be 'yes' because the poorest person is unambiguously better off ($\mathbf{y}^1_{1\cdot} > \mathbf{x}^1_{1\cdot}$) and the richest person is unambiguously worse off ($\mathbf{y}^1_{4\cdot} < \mathbf{x}^1_{4\cdot}$). Inequality is certainly lower in $\mathbf{Y}^1$ than in $\mathbf{X}^1$.

Can we thus state that multidimensional inequality would be lower whenever one joint distribution is obtained from another by UBT (or UPDT)? The answer is not straightforward. Let us consider another example motivated by Dardanoni (1996), where $\mathbf{Y}^2$ is obtained from $\mathbf{X}^2$, such that $\mathbf{Y}^2 = \mathbf{B}\mathbf{X}^2$:

$$\mathbf{Y}^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 7 & 7 \\ 7 & 7 \end{bmatrix}; \mathbf{X}^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 5 & 9 \\ 9 & 5 \end{bmatrix} \text{and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}.$$

In this case, the achievements of the two richest persons were averaged, while the achievements of the two poorest persons remained unchanged. Clearly, $\mu_j(\mathbf{Y}^2) = \mu_j(\mathbf{X}^2)$ for $j = 1,2$. Suppose each person's human development is obtained by aggregating her achievements using an aggregation function: $f(x^2_{i1}, x^2_{i2}) = (x^2_{i1}x^2_{i2})^{0.5}$, which is a standard concave Cobb-Douglas function. The human development levels for the two poorest persons remain unchanged, i.e., $f(y^2_{i1}, y^2_{i2}) = f(x^2_{i1}, x^2_{i2})$ for $i = 1,2$. However, the human development levels are certainly higher for the two richest persons, i.e., $f(y^2_{i1}, y^2_{i2}) > f(x^2_{i1}, x^2_{i2})$ for $i = 3,4$. What we see is that a reduction in inequality within both marginal distributions uniformly in $\mathbf{Y}^2$ has made the two richest persons better off but has left the two poorest persons behind.

Can it thus be claimed that inequality is lower in $\mathbf{Y}^2$ than in $\mathbf{X}^2$? There are two major issues with such comparisons. One is that the transfer is restricted to occur uniformly across all dimensions, which may not be reasonable in practice. Second, transfers do not necessarily take place between a richer person and a poorer person, which is the main essence of the Pigou-Dalton transfer in the unidimensional context (Lasso de la Vega et al., 2010). For UPDT and UBT, transfers may take place between two persons where one has higher achievements in some dimensions but lower achievements than the other person in other dimensions.

Through a novel approach, Bosmans et al. (2015) provide an explanation for the comparison between $\mathbf{Y}^2$ and $\mathbf{X}^2$ by decomposing overall multidimensional inequality into an **inequity** component and an **inefficiency** component. Inequity within a joint distribution exists as long as all persons do not have the mean achievement within each dimension. Inefficiency within a joint distribution exists insofar as the wellbeing level of at least one person can be improved through redistribution without worsening the wellbeing levels of any other person. The redistribution between $\mathbf{Y}^2$ and $\mathbf{X}^2$ have indeed increased inequity in wellbeing by leaving the two poorest behind (similar point was raised by Duclos et al., 2011, p. 229).

However, the wellbeing levels of the two richest persons have increased without worsening anyone else's wellbeing level, improving efficiency. The improvement in efficiency may have outweighed the deterioration in inequity, leading to a net improvement in inequality. Bosmans et al. (2015) thus conclude that 'uniform majorization is more successful at capturing the efficiency aspect of multidimensional inequality than at capturing the equity aspect' (p. 99).

Fleurbaey and Trannoy (2003) have proposed another extension of the Pigou-Dalton progressive transfer in the multidimensional context referred to as the **Pigou-Dalton Bundle Transfer** (PBT). The transfer is rank preserving and takes place only between a rich person and an unambiguously poorer person. A person h is unambiguously richer than another person k whenever $\mathbf{x}_{h\cdot} > \mathbf{x}_{k\cdot}$. In $\mathbf{X}^2$, for example, the first two persons are unambiguously poorer than the last two persons. The **PBT majorization** requires inequality to be lower if a distribution is obtained from another distribution by a PBT or a sequence of PBTs.

Building on Lasso de la Vega et al. (2010), for any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a PBT whenever there are two persons, h and k, such that (i) $\mathbf{x}_{h\cdot} > \mathbf{x}_{k\cdot}$, (ii) $\mathbf{y}_{k\cdot} = \mathbf{x}_{k\cdot} + \delta$ and $\mathbf{y}_{h\cdot} = \mathbf{x}_{h\cdot} - \delta$ for some $\delta = (\delta_1, \dots, \delta_d) > 0$, (iii) $\mathbf{y}_{i\cdot} = \mathbf{x}_{i\cdot}$ for all $i \neq h, k$, and (iv) $\mathbf{y}_{h\cdot} \geq \mathbf{y}_{k\cdot}$. What do all these conditions mean? Condition (i) requires that person h has a higher achievement than person k in at least one dimension and no less achievement in any dimension before transfer. Condition (ii) requires that achievement(s) of a positive amount in at least one dimension is transferred from person h to person k. Condition (iii) requires that achievements of all other persons are identical in $\mathbf{Y}$ and $\mathbf{X}$. Finally, condition (iv) requires that the post-transfer achievements of person h are not lower than that of person k in any dimension, ensuring that post-transfer ranks are preserved.

In the following example, $\mathbf{Y}^3$ is obtained from $\mathbf{X}^3$ by a PBT:

$$\mathbf{Y}^3 = \begin{bmatrix} 2 & 4 \\ 3 & 4 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{X}^3 = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 6 & 7 \\ 9 & 5 \end{bmatrix} \text{and } \delta = (1,2).$$

The first person in $\mathbf{X}^3$ is poorer than all others in both dimensions. An achievement of one unit in dimension 1 ($\delta_1 = 1$) and an achievement of two units in dimension 2 ($\delta_2 = 2$) are transferred from person 2 to person 1. Importantly, after the transfer (i.e., in $\mathbf{Y}^3$), in no dimension are person 1's achievements larger than that of person 2, yet $\mu_j(\mathbf{Y}^3) = \mu_j(\mathbf{X}^3)$ for all j. Inequality $\mathbf{Y}^3$ may surely be claimed to be lower than that in $\mathbf{X}^3$ as required by the PBT majorization.[21]

So far, we have presented extensions of Pigou-Dalton transfers. Although these transfers can rank some joint distributions, they may not be able to rank several others. In the multidimensional context, there is another form of inequality that is not relevant at all to the unidimensional context. This second form is concerned with association between dimensions. Let us consider the following example involving $\mathbf{Y}^4$ and $\mathbf{X}^4$ to clarify the point:

$$\mathbf{Y}^4 = \begin{bmatrix} 1 & 10 \\ 2 & 9 \\ 8 & 3 \\ 9 & 2 \end{bmatrix}; \mathbf{X}^4 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 8 & 9 \\ 9 & 10 \end{bmatrix}.$$

Note that each marginal distribution in $\mathbf{Y}^4$ is identical to the respective marginal distribution in $\mathbf{X}^4$, i.e., $\mathbf{y}_{\cdot 1}^4 = \mathbf{x}_{\cdot 1}^4 = (1,2,8,9)$ and $\mathbf{y}_{\cdot 2}^4 = \mathbf{y}_{\cdot 2}^4 = (2,3,7,8)$ and certainly $\mu_j(\mathbf{Y}^4) = \mu_j(\mathbf{X}^4)$ for all j. The main difference between the two joint distributions is observed when we look at the marginal distributions together within each joint distribution. In $\mathbf{X}^4$, both marginals are similarly ordered or are perfectly positively associated with each other; each poorer person is poorer in both dimensions than each richer person. Contrarily, in $\mathbf{Y}^4$, marginals are oppositely ordered or are perfectly negatively associated with each other. Which distribution is more equal: $\mathbf{Y}^4$ or $\mathbf{X}^4$?

In their pioneering paper, Atkinson and Bourguignon (1982) introduced a different form of multidimensional inequality that is concerned with the joint distribution of achievements, requiring multidimensional social evaluations to be sensitive to correlation, or, more precisely, association, between dimensions. They derived mathematical conditions that allow ranking bivariate joint distributions when they have the same marginal distributions, but different interdependence, just like in the example involving $\mathbf{Y}^4$ and $\mathbf{X}^4$**.** Decancq (2012) has extended the Atkinson-Bourguignon framework to situations involving three or more dimensions.

In the literature of multidimensional measurement, certain approaches have been proposed to capture sensitivity to change in the interdependence between multiple dimensions. One prominent approach is the **Correlation Increasing Transfer***,* coined by Tsui (1999) and motivated by Boland and Proschan (1988). The concept is also variously referred to as Correlation Increasing Switch (Bourguignon and Chakravarty, 2003), Correlation Increasing Rearrangement (Deutsch and Silber, 2005), and Association

21 We have not taken the preferences of persons into consideration. For discussions on how transfers such as PBT and UM may become incompatible when one allows individual preferences to differ, see Fleurbaey and Trannoy (2003), Trannoy (2006), and Decancq et al. (2015).

Increasing Transfer (Seth 2013). We explain the concept using an example with $\mathbf{Z}^5$, $\mathbf{Y}^5$, and $\mathbf{X}^5$ involving three dimensions.

$$\mathbf{Z}^5 = \begin{bmatrix} 1 & \mathbf{3} & 2 \\ 2 & \mathbf{4} & 3 \\ 8 & 8 & 6 \\ 9 & 9 & 9 \end{bmatrix}; \mathbf{Y}^5 = \begin{bmatrix} 1 & \mathbf{4} & 2 \\ 2 & \mathbf{3} & 3 \\ 8 & \mathbf{8} & \mathbf{6} \\ 9 & \mathbf{9} & \mathbf{9} \end{bmatrix}; \mathbf{X}^5 = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 3 \\ 8 & \mathbf{9} & \mathbf{9} \\ 9 & \mathbf{8} & \mathbf{6} \end{bmatrix}$$

If we compare the joint distributions, then clearly their marginals are identical: $\mathbf{z}_{\cdot 1}^5 = \mathbf{y}_{\cdot 1}^5 = \mathbf{x}_{\cdot 1}^5 = (1,2,8,9)$, $\mathbf{z}_{\cdot 2}^5 = \mathbf{y}_{\cdot 2}^5 = \mathbf{x}_{\cdot 2}^5 = (3,4,8,9)$, and $\mathbf{z}_{\cdot 3}^5 = \mathbf{y}_{\cdot 3}^5 = \mathbf{y}_{\cdot 3}^5 = (2,3,6,9)$. In $\mathbf{X}^5$, marginals are not perfectly positively associated. For example, the fourth person has a higher achievement than the third person in the first dimension but lower achievements in other dimensions. Their achievements are swapped among themselves to obtain $\mathbf{Y}^5$ so that the fourth person has higher achievements in all three dimensions than the third person. Clearly, the association between dimensions is higher in $\mathbf{Y}^5$. Next, the achievements between the first two persons in $\mathbf{Y}^5$ are further swapped among themselves to obtain $\mathbf{Z}^5$, increasing the association further. All three dimensions in $\mathbf{Z}^5$ are perfectly positively associated.

Formally, for any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an **association increasing transfer** (AIT) if there are two persons h and k, such that (i) $\mathbf{y}_{hj} = \min\{\mathbf{x}_{hj}, \mathbf{x}_{kj}\}$ and $\mathbf{y}_{kj} = \max\{\mathbf{x}_{hj}, \mathbf{x}_{kj}\}$ for all j, (ii) $\mathbf{y}_{i\cdot} = \mathbf{x}_{i\cdot}$ for all $i \neq h, k$, and (iii) $\mathbf{Y}$ is not a permutation of $\mathbf{X}$.[22] The third condition in the definition is important as it prevents the transfer from taking place in dimensions where both persons have equal achievements. If $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an association increasing transfer then, conversely, $\mathbf{X}$ is stated to be obtained from $\mathbf{Y}$ by an **association decreasing transfer** (ADT).

An **association increasing majorization** requires that inequality should increase when one distribution is obtained from another by an AIT or a sequence of AITs. A **converse association increasing majorization** requires that inequality should fall when one distribution is obtained from another by an AIT or a sequence of AITs.

Should multidimensional inequality increase or decrease due to an association-increasing transfer? Tsui (1999), among others, requires inequality to increase (or at least not to decrease) whenever there is an AIT. Implicitly, this requirement assumes dimensions to be substitutes (Atkinson and Bourguignon, 1982; Bourguignon and Chakravarty, 2003). If a good health outcome can *substitute* for a low income or a low educational level, it is preferred that high achievements are spread out across the population rather than having a few people with high achievements in all dimensions. However, if attributes are *complements* – say,

22 We prefer to use the term 'association' rather than the term 'correlation', as correlation merely encompasses linear relationship between variables.

if a good health outcome is necessary to achieve higher income or better education, an AIT may produce a preferable distribution. This relationship, however, has recently been questioned by Bosmans et al. (2015), who, with a specific example, show that association-increasing majorization may be compatible with complementarity. This controversial topic however requires further research.

Another approach to capturing sensitivity to change in the interdependence between multiple dimensions has been proposed by Dardanoni (1996), which Decancq and Lugo (2012b) refer to as **unfair rearrangement** (UR). For any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an UR if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a sequence of AITs such that there is vector dominance between every pair of persons in $\mathbf{Y}$. In words, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an UR, then one individual has the largest achievements in all dimensions, another individual has the second largest achievements in all dimensions, and so on. In our example, $\mathbf{Z}^5$ is obtained from both $\mathbf{Y}^5$ and $\mathbf{X}^5$ by UR. Note, however, that UR cannot rank $\mathbf{Y}^5$ and $\mathbf{X}^5$. Should inequality increase due to unfair rearrangement? Similar controversy may arise as in case of the association-increasing majorization and converse association-increasing majorization.

Finally, we discuss the concept of **Compensating Transfer** (CT), proposed by Lasso de la Vega et al. (2010), which combines the concept of PBT and ADT. For any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a CT whenever there are two persons, h and k, such that (i) $\mathbf{x}_{h\cdot} > \mathbf{x}_{k\cdot}$, (ii) $\mathbf{y}_{k\cdot} = \mathbf{x}_{k\cdot} + \delta$ and $\mathbf{y}_{h\cdot} = \mathbf{x}_{h\cdot} - \delta$ for some $\delta = (\delta_1, \dots, \delta_d) > 0$, (iii) $\mathbf{y}_{i\cdot} = \mathbf{x}_{i\cdot}$ for all $i \neq h, k$, and (iv) $\mathbf{y}_{h\cdot} \geq \mathbf{x}_{k\cdot}$. The definition of CT is analogous to the definition of PBT, but with one crucial difference. The fourth condition in the definition of CT allows reversal of ranks by requiring $\mathbf{y}_{h\cdot} \geq \mathbf{x}_{k\cdot}$, unlike $\mathbf{y}_{h\cdot} \geq \mathbf{y}_{k\cdot}$, in case of a PBT. A CT is claimed to lower multidimensional inequality.

In the following example, $\mathbf{Y}^6$ is obtained from $\mathbf{X}^6$ by a CT.

$$\mathbf{Y}^6 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{X}^6 = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 6 & 7 \\ 9 & 5 \end{bmatrix} \text{ and } \delta = (3,1).$$

The first person in $\mathbf{X}^6$ is poorer than all others. An achievement of three units in dimension 1 ($\delta_1 = 3$) and an achievement of one unit in dimension 2 ($\delta_2 = 1$) are transferred from person 2 to person 1. After the transfer (i.e., in $\mathbf{Y}^6$), person 1's achievement in the first dimension is higher than that of person 2, but person 1's achievement in the second dimension remains lower. In this case, therefore, there is no vector dominance between persons 1 and 2. The CT from $\mathbf{X}^6$ to $\mathbf{Y}^6$ can be broken down into a PBT (from $\mathbf{X}^6$ to $\mathbf{Z}^6$ by δ') and an ADT (from $\mathbf{Z}^6$ to $\mathbf{Y}^6$) as

$$\mathbf{Y}^6 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{Z}^6 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{X}^6 = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 6 & 7 \\ 9 & 5 \end{bmatrix} \text{ and } \delta' = (1,1).$$

Our detailed discussions in this section thus explained the difficulties that one may face when assessing inequality within a multidimensional framework.

4. Multidimensional Inequality Measures in a Normative Framework

In this section, we discuss various normative multidimensional inequality measures that have been proposed in the literature. Inequality within a single dimension $\mathbf{x} \in \mathbb{R}^n_{++}$ (using notation from Section 2) is assessed by defining a unidimensional inequality measure, which is a function $I(\mathbf{x})$ that maps the achievements in $\mathbf{x}$ in a real number $\mathbb{R}$. Technically, $I: \mathbb{R}^n_{++} \to \mathbb{R}$. In the unidimensional context, inequality is exclusively about *inequality across people* within a certain dimension, say income. Similarly, in the multidimensional context, an inequality measure maps from the achievements in $\mathbf{X}$ to a real number $\mathbb{R}$, i.e., $I: \mathcal{X} \to \mathbb{R}$.

In Section 3, while discussing the challenges of multidimensional poverty assessment, we already introduced a set of distributional properties, namely, those related to transfer and those related to association across dimensions. In the next subsection, we introduce some of the important non-distributional properties.

4.1 Non-Distributional Properties

Normative inequality measures are required to satisfy certain normative properties for improving the structure of the measures – either to make them more amicable to empirical applications or to make the assessment technically sound.[23] We have already discussed various majorization principles in Section 3. Inequality measures are required to satisfy certain additional properties – some of which are basic whereas others are more controversial.

Let us first discuss the basic properties. One basic property, **symmetry** (also called anonymity), requires that an inequality measure should be invariant to who has each achievement vector. Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by simply permuting the achievement vectors in $\mathbf{X}$ (i.e., $\mathbf{Y} = \mathbf{PX}$ where $\mathbf{P}$ is a permutation matrix), then $I(\mathbf{X}) = I(\mathbf{Y})$. A second basic property, **replication invariance** (also called the population principle), requires that an inequality measure should be invariant to replications of

23 For detailed presentations and explanations of the normative properties in the unidimensional context, see Foster et al. (2013).

the population. Technically, if $\mathbf{Y} \in \mathcal{X}_{\gamma n}$ is obtained from $\mathbf{X} \in \mathcal{X}_n$ by simply replicating or cloning each person's achievement vector in $\mathbf{X}$ by γ times (where γ is a positive integer and $\gamma > 1$), then $I(\mathbf{X}) = I(\mathbf{Y})$. This property allows comparing inequality across societies with different population sizes. A third basic property, **normalisation**, requires that whenever every person has the same achievement vector the inequality measure should be equal to zero. Technically, for any $\mathbf{X} \in \mathcal{X}_n$, if $\mathbf{x}_{i\cdot} = \mathbf{x}_{k\cdot}$ for all $i \neq k$, then $I(\mathbf{X}) = 0$.

The next set of normative properties are slightly controversial and not all are necessarily compatible with each other. The **scale invariance** (also called zero-degree homogeneity) property requires that an inequality measure should be invariant to proportional changes in all achievements. Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ such that $\mathbf{Y} = \delta\mathbf{X}$ for any $\delta > 0$, then $I(\mathbf{X}) = I(\mathbf{Y})$. A more intuitive but related property is **ratio scale invariance**, which requires that if each column vector is multiplied by a factor, then this should not alter the level of inequality. Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ such that $\mathbf{y}_{\cdot j} = \delta_j \mathbf{x}_{\cdot j}$ for any $\delta_j > 0$ for all j, then $I(\mathbf{X}) = I(\mathbf{Y})$. The ratio scale invariance property allows comparing the level of inequality when achievements are presented in alternative units (e.g., income may be assessed by US dollars or British Pounds, education may be assessed in years or months). A third related property, but with a weaker requirement than the ratio scale invariance property, is unit consistency, which merely requires that the ordering of distributions by an inequality measure should not alter whenever units of measurement change (Zheng, 2007). Suppose for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $I(\mathbf{X}) > I(\mathbf{Y})$. If $\mathbf{Y}' \in \mathcal{X}_n$ is obtained from $\mathbf{Y}$ and $\mathbf{X}' \in \mathcal{X}_n$ is obtained from $\mathbf{X}$ so that $\mathbf{y}'_{\cdot j} = \delta_j \mathbf{y}_{\cdot j}$ and $\mathbf{x}'_{\cdot j} = \delta_j \mathbf{x}_{\cdot j}$ for any $\delta_j > 0$ for all j, then $I(\mathbf{X}') > I(\mathbf{Y}')$. Finally, the **translation invariance** property requires that an inequality measure should be invariant when all achievements within each distribution are changed by certain amounts (Kolm, 1976). Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n_{++}$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ such that $y_{ij} = x_{ij} + \delta_j$ for any $\delta_j \in \mathbb{R}$ for all j and for all i, then $I(\mathbf{X}) = I(\mathbf{Y})$.

Along with other properties, an inequality measure that satisfies scale invariance or ratio scale invariance or unit consistency is referred to as a **relative inequality measure**. An inequality measure that satisfies translation invariance, along with other properties, is referred to as an absolute inequality measure. It should be noted that no inequality measure can simultaneously be relative and absolute. Kolm (1976) refers to the relative viewpoint as rightist and the absolute viewpoint as leftist. In this paper, we focus on relative multidimensional inequality measures.

Finally, in the assessment of inequality, it is often required to understand the link between overall inequality and the inequality of different population subgroups whenever the entire population is divided into a collection of mutually exclusive and collectively exhaustive subgroups. For example, the entire population of a country may be subgrouped across states or provinces or subgrouped across ethnic or religious groups. The **subgroup consistency** property requires that an increase in inequality in one subgroup should lead to an increase in overall inequality if inequality within other subgroups remains unchanged. The **subgroup decomposability** property requires that overall inequality can be expressed in terms of the inequality levels of subgroups, their vector of average achievements in different dimensions, and their population sizes.

4.2 Structure

We now turn to the structure of inequality measures. Foster (2008) eloquently showed that – except for some limiting cases – all unidimensional inequality measures can be presented as a function of two achievement standards, where an **achievement standard** is a measure of the size of a distribution of achievement.[24] Some of the achievement standards may be viewed as social welfare functions (W). In those cases, an inequality index can be presented as a function of two social evaluation functions: $W(\mathbf{x})$ and $W(\bar{\mathbf{x}})$, where $\bar{\mathbf{x}}$ is obtained from $\mathbf{x}$ such that $\bar{\mathbf{x}} = \big(\mu(\mathbf{x}), \dots, \mu(\mathbf{x})\big)$ or each person in $\bar{\mathbf{x}}$ receives the average achievement $\mu(\mathbf{x})$. An inequality measure is typically presented either as $I(\mathbf{x}) = [W(\bar{\mathbf{x}}) - W(\mathbf{x})]/W(\bar{\mathbf{x}})$ whenever $W(\bar{\mathbf{x}}) > W(\mathbf{x})$ or $I(\mathbf{x}) = [W(\mathbf{x}) - W(\bar{\mathbf{x}})]/W(\mathbf{x})$ whenever $W(\bar{\mathbf{x}}) < W(\mathbf{x})$.

Analogously, in the multidimensional context, almost all multidimensional inequality measures are functions of two social evaluation functions: $W(\mathbf{X})$ and $W(\bar{\mathbf{X}})$, where $\bar{\mathbf{X}}$ is obtained from $\mathbf{X}$ such that $\bar{\mathbf{x}}_{i\cdot} = \boldsymbol{\mu}(\mathbf{X})$ for all $i = 1, \dots, n$ or each person in $\bar{\mathbf{X}}$ receives the mean achievement in all d dimensions. Similarly, the typical approach to present an inequality measure is

$$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}. \tag{1}$$

Unlike in the unidimensional framework, multidimensional social welfare functions are constructed using a two-step aggregation approach.[25] Some of them pursue a **row-first** aggregation approach; whereas others pursue a **column-first** aggregation approach. A row-first aggregation approach uses an aggregation function in the first step to aggregate the achievements of each person (row of $\mathbf{X}$) in all d dimensions to obtain an **aggregate individual achievement**; then all n aggregate individual achievements are aggregated using another aggregation function to obtain the overall **social evaluation**. A column-first aggregation approach, on the other hand, uses an aggregation function in the first step to aggregate the

24 The limiting cases are one of Theil's measures and the variance of logarithms. Foster (2006) uses the term 'income standard' to refer to the size of any unidimensional income distribution as opposed to the term 'achievement standard' that we use here.

25 See Bosmans et al. (2015) for an axiomatic justification of the structure in (1) as well as the two-step aggregation approach.

achievements in each dimension (column of $\mathbf{X}$) of all n persons to obtain an **aggregate dimensional achievement**; then all d aggregate dimensional achievements are aggregated using another aggregation function to obtain the overall social evaluation. Given that the column-first aggregation function first aggregates achievements across each dimension, it is not possible to capture the second form of multidimensional inequality that reflects association between dimensions.[26]

4.3 Measures

In Table 1, we summarise the social evaluation functions of various multidimensional inequality measures proposed in the literature. Interestingly, all the social evaluation functions in Table 1 use either a generalised mean evaluation function or a generalised Gini evaluation function in either step. For $m \geq 2$, for any $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m_{++}$ and for any $\mathbf{w} = (w_1, \dots, w_m) \in \mathbb{R}^m_{+}$ such that $\mathbf{w} \geq 0$ and $\sum_{k=1}^m w_k = 1$, the generalised mean of order $\beta \in \mathbb{R}$ is defined as

$$GM(\mathbf{a}; \mathbf{w}, \beta) = \begin{cases} \left(\sum_{k=1}^{m} w_k a_k^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \neq 0 \\ \prod_{k=1}^{m} a_k^{w_k} & \text{for } \beta = 0 \end{cases}. \quad (2)$$

The expression for a general mean is also equivalent to the expression of the **constant elasticity of substitution function**. The generalised mean evaluation function for $\beta < 1$ is used to construct Atkinson's well-known unidimensional inequality measure (Atkinson 1970).[27]

For $m \geq 2$ and for any $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m_{++}$ the generalised Gini evaluation function is defined as

$$GG(\mathbf{a}; \delta) = \sum_{k=1}^{m} \left[\left(\frac{r_k}{m}\right)^{\delta} - \left(\frac{r_k - 1}{m}\right)^{\delta}\right] a_k; \quad (3)$$

where r_k is the rank of the k^{th} element in $\mathbf{a}$ when all elements are ranked in descending order. Note that $GG(\mathbf{a}; \delta)$ is also a type of average, where the k^{th} element is assigned a weight of $w_k' = (r_k/m)^{\delta} - ([r_k - 1]/m)^{\delta}$. It is straightforward to verify that $\sum_{k=1}^m w_k' = 1$. In this evaluation function, smaller

26 For a class of standard of living measures that are invariant to the order of aggregation, see Dutta et al. (2003). If one requires a social evaluation function to provide the same evaluation irrespective of whether a row-first or a column-first aggregation is used, then the evaluation function becomes too restrictive.

27 The generalised mean takes different forms for different values of β. For $\beta = 1$, it is the arithmetic mean; for $\beta = 0$, it is the geometric mean; for $\beta = -1$, it is the harmonic mean; and for $\beta = 2$, it is the Euclidean mean. When $\beta > 1$, a larger weight is placed on larger elements of $\mathbf{a}$ and the general mean approaches its maximum element as $\beta \to \infty$. For $\beta < 1$, a larger weight is placed on lower elements of $\mathbf{a}$ and the general mean approaches its minimum element as $\beta \to -\infty$.

elements receive larger weights.[28] Setting $\delta = 2$ in equation (3), we obtain the Gini social evaluation function:

$$GG(\mathbf{a};2) = \sum_{k=1}^{m}\left[\frac{(2i-1)}{m^2}\right]a_k. \quad (4)$$

The Gini social evaluation function is used to compute the well-known Gini coefficient.

We should point out that a pioneering multidimensional inequality measure proposed by Maasoumi (1986) differs from equation (1) not only in general structure, but also in the distribution that is considered as the most equal. Maasoumi (1986) used a row-first aggregation approach, but a key difference from other measures is that the most equal distribution is one in which the aggregate individual achievements are equal and not necessarily when everyone has the equal achievement vector. Consider the following achievement matrices:

$$\mathbf{Y}' = \begin{bmatrix} 3 & 7 \\ 3 & 7 \\ 7 & 3 \\ 7 & 3 \end{bmatrix} \text{ and } \mathbf{X}' = \begin{bmatrix} 5 & 5 \\ 5 & 5 \\ 5 & 5 \\ 5 & 5 \end{bmatrix}.$$

Suppose, the level of human development for each individual is assessed by $U(x'_{i1}, x'_{i2}) = (x'_{i1}x'_{i2})^{0.5}$. Massoumi (1986) would consider both $\mathbf{Y}'$ and $\mathbf{X}'$ to be the most egalitarian; whereas other normative inequality measures would consider only $\mathbf{X}'$ to be the most egalitarian.

It is worth noting that all the inequality measures detailed in Table 1 are sensitive to distribution. In other words, they satisfy UPD majorization or uniform majorization. Additionally, the measures proposed by Bourguignon (1999), Tsui (1995, 1999), Decancq and Lugo (2012b), Seth (2013) and Diez et al. (2007) are sensitive to association between dimensions under appropriately selected parameter restrictions. However, Gajdos and Weymark (2005) and Foster et al. (2005), as they use column-first aggregation, are not sensitive to association between dimensions. In fact, measures proposed by Foster et al. (2005) yield the same level of social evaluation whether a row-first or a column-first aggregation is used. In this case, the social evaluation function may be referred to as **path independent**. All measures presented in Table 1 require the variables under consideration to be cardinal.

28 The weight assigned to the largest element is $(1/m)^\delta - (0/m)^\delta$ or $1/m^\delta$; whereas the weight assigned to the smallest element is $(m/m)^\delta - ([m-1]/m)^\delta = 1 - ([m-1]/m)^\delta$.

Table 1: Multidimensional Inequality Measures, Relevant Social Evaluation Functions and the Order of Aggregation

	Order of aggregation	First stage aggregation function	Second stage aggregation function	Inequality measure
1. Tsui (1995)	Row-first	$h_i = \prod_{j=1}^{d} x_{ij}^{\alpha_j}$	$W(\mathbf{X}) = \begin{cases} \left[\frac{1}{n}\sum_{i=1}^{n} h_i\right]^{\frac{1}{\Sigma_{j=1}^{d}\alpha_j}} \\ \left[\prod_{i=1}^{n} h_i^{\frac{1}{\Sigma_{j=1}^{d}\alpha_j}}\right]^{\frac{1}{n}} \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
2. Bourguignon (1999)	Row-first	$h_i = \begin{cases} \left(\sum_{j=1}^{d} w_j x_{ij}^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta < 1 \;\&\; \beta \neq 0 \\ \prod_{j=1}^{d} x_{ij}^{w_j} & \text{for } \beta = 0 \end{cases}$	$W(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} h_i^{\alpha}\,;\, 0 < \alpha < 1$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
3. Foster, Lopez-Calva and Szekely (2005)*	Column-first	$h_j = \begin{cases} \left(\sum_{i=1}^{n} x_{ij}^{\alpha}\right)^{\frac{1}{\alpha}} & \text{for } \alpha < 1 \;\&\; \alpha \neq 0 \\ \left(\prod_{i=1}^{n} x_{ij}\right)^{\frac{1}{n}} & \text{for } \alpha = 0 \end{cases}$	$W(\mathbf{X}) = \begin{cases} \left(\sum_{j=1}^{d} w_j h_j^{\alpha}\right)^{\frac{1}{\alpha}} & \text{for } \alpha < 1 \;\&\; \alpha \neq 0 \\ \prod_{j=1}^{d} h_j^{w_j} & \text{for } \alpha = 0 \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
4. Gajdos and Weymark (2005)**	Column-first	$h_j = \sum_{i=1}^{n}\left[\left(\frac{r_i^j}{n}\right)^{\delta} - \left(\frac{r_i^j - 1}{n}\right)^{\delta}\right] x_{ij}$; $\delta > 0$ & r_i^j is the rank of person i in dimension j	$W(\mathbf{X}) = \begin{cases} \left(\sum_{j=1}^{d} w_j h_j^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \neq 0 \\ \prod_{j=1}^{d} h_j^{w_j} & \text{for } \beta = 0 \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
5. Decancq and Lugo (2012b)	Row-first	$h_i = \begin{cases} \left(\sum_{j=1}^{d} w_j x_{ij}^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \neq 0 \\ \prod_{j=1}^{d} x_{ij}^{w_j} & \text{for } \beta = 0 \end{cases}$	$W(\mathbf{X}) = \sum_{i=1}^{n}\left[\left(\frac{r_i}{n}\right)^{\delta} - \left(\frac{r_i - 1}{n}\right)^{\delta}\right] h_i$; $\delta > 0$ & r_i is the rank of person i in dimension j	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$

	Order of aggregation	First stage aggregation function	Second stage aggregation function	Inequality measure
6. Seth (2013)	Row-first	$h_i = \begin{cases} \left(\sum_{j=1}^{d} w_j x_{ij}^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \leq 1 \,\&\, \beta \neq 0 \\ \prod_{j=1}^{d} x_{ij}^{w_j} & \text{for } \beta = 0 \end{cases}$	$W(\mathbf{X}) = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} h_i^{\alpha}\right)^{\frac{1}{\alpha}} & \text{for } \alpha \leq 1 \,\&\, \alpha \neq 0 \\ \left(\prod_{j=1}^{n} h_i\right)^{\frac{1}{n}} & \text{for } \alpha = 0 \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
7. Tsui (1999) and Diez et al. (2007)	Row-first	$h_i = \prod_{j=1}^{d} x_{ij}^{\alpha_j}$	$W(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} h_i$	$I(\mathbf{X}) = \phi\left(\rho \prod_{j=1}^{d} \mu_j^{\tau}\left[\frac{W(X)}{W(\bar{X})} - 1\right]\right)$
8. Tsui (1999)	Row-first	$h_i = \sum_{j=1}^{d} \delta_j \log(x_{ij})$	$W(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} h_i$	$I(\mathbf{X}) = \phi\left(\rho\left[\frac{W(X)}{W(\bar{X})} - 1\right]\right)$

* Both stages of aggregation use the same parameter α. This social evaluation function is path independent.

** Here we present only one measure from the class of indices proposed by Gajdos and Wermark (2005) based on a Gini social evaluation function. The first stage the aggregation function may be a *generalised* Gini social evaluation function $h_j(x_{\cdot j}) = \sum_{i=1}^{n} a_i\, x_{ij}^{Ord}$ such that $0 < a_1 < a_2 < \cdots < a_n$ and $\sum_{i=1}^{n} a_i = 1$.

5. Empirical Applications

Many of the indices presented in Table 1 of Section 4 have been used to measure inequality in human development.[29] An early inequality-adjusted index of human development is the Gender-related Development Index (GDI), which was proposed by Anand and Sen (1995) and has been annually reported in HDRs between 1995 and 2009. The GDI aims at capturing inequality across gender in the same set of dimensions (health, knowledge and living standard) as the well-known Human Development Index (HDI). The construction of the GDI is based on column-first aggregation. In the first step, an equally distributed equivalent (EDE) achievement (using a harmonic mean, $\beta = -1$ in Equation (1)) is obtained by the aggregating achievements of males and females in each dimension. In the second step, the GDI is obtained as an average of the three-dimensional EDE achievements.

The GDI is aimed at making the assessment of human development sensitive to inequality between genders. Hicks (1997), by contrast, proposes an inequality-adjusted HDI to capture inequality across the population. Hicks also uses a column-first aggregation, first computing a Gini coefficient-adjusted achievement for each dimension (in a spirit similar to Gajdos and Weymark (2005)) and then averaging these three Gini coefficient-adjusted achievements to obtain the overall index.[30] Due to data availability constraints, the Gini coefficient for living standard is computed using quintile income shares of income; for knowledge it is computed using shares of the population across six ordered categories of education; and for health it is computed using mortality statistics across age, gender and area of residence.

Building upon Anand and Sen (1995) and Hicks (1997), Foster et al. (2005) propose a family of distribution-sensitive HDIs, which is invariant to the order to aggregation (reported in Table 1). Unlike its two predecessors, the Foster et al. indices are subgroup consistent (the concept is explained in Section 4). Foster et al. (2005) used their indices to study the link between the level of and the inequality in human development in Mexico using a sample of the 2000 population census data. Due to the lack of individual/household-level health information, they were only able to capture inequality in that dimension across municipalities by using infant survival rates. They use per capita household income to assess living standard and take a weighted average of literacy and attendance information to assess knowledge at the household level. The empirical results exemplify the relevance of considering sensitivity to inequality. When sensitivity to inequality is considered, Mexican states' rankings change considerably when compared to when inequality is ignored. A particular index with $\alpha = 0$ from their family of indices has been used to construct UNDP's inequality-adjusted HDI (UNDP, 2010; Alkire and Foster, 2010).

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29 For a review of inequality-sensitive indices of human development, see Seth and Villar (2017ab).

30 The Gini coefficient-adjusted achievement of a dimension is computed as $\mu(1 - G)$, where μ is the average dimensional achievement and G is the dimensional Gini coefficient.

The inequality-adjusted indices proposed by Anand and Sen (1995), Hicks (1997) and Foster et al. (2005) represent very significant progress in incorporating one form of inequality. In the case of the GDI, though, this is limited to inequality only between gender. These indices are, however, not sensitive to the joint distribution of achievements, because the indices proposed by Anand and Sen (1995) as well as Hicks (1997) use column-first aggregation and the Foster et al. (2005) family of indices are invariant to the order of aggregation.[31] When one requires the assessment to be sensitive to joint distributions, one needs to decide upon the normative association properties discussed in Section 4. There are strong arguments in favour of using indices sensitive to the joint distribution of achievements (Seth 2009, 2013; Decancq 2017; Seth and Villar 2017a).

Seth (2009, 2013), building on Foster et al. (2005), proposes a family of wellbeing indices by row-first aggregation that are sensitive to both forms of multidimensional inequality. These indices, at the first stage, aggregate the achievements of each person across all dimensions using a generalised mean of order β, and then in the second stage aggregate individual achievements of a general mean of order α. Interestingly, the restriction that $\alpha \neq \beta$ makes these indices sensitive to the joint distribution of achievements.[32] Seth (2009) applies these indices to the same Mexican dataset studied by Foster et al. (2005); whereas Seth (2013) uses these indices to study the change in the level of human development in Indonesia between 1997 and 2000, using the Indonesian Family Life Survey panel datasets. The exercise with Mexico shows how reflecting sensitivity to inequality across dimensions, unlike Foster et al. (2005), further changes the estimated level of human development as well as the rankings for several Mexican states. Seth's (2013) study of Indonesia found an interesting result: while association between the considered dimensions increased between 1997 and 2000, inequality within the distribution of monetary achievements decreased significantly. When an index insensitive to the joint distribution of achievements is used, no significant change in human development is observed. However, for certain combinations of parameter values, the level of human development reflects a statistically significant increase, showing that the decrease in inequality within the monetary dimension offsets the increase in association in the joint distribution of achievements.

Analysing the cross-country results based on the GDI from various HDRs, as well as analysing the findings in Hicks (1997), Foster et al. (2005) and Seth (2009), Seth and Villar (2017b) observe a consistent inverse relationship between the level of human development and inequality in human development, both across

31 An advantage of the Foster et al. (2005) family of indices is that they are path-independent. This means that both the row-first and column-first aggregations yield the same value whenever data on all dimensions are available at the same disaggregated level. The indices are also convenient when micro-data are not available for all dimensions at the same disaggregated level.

32 The Foster et al. (2005) family of indices is a subfamily of the Seth family of indices, when $\alpha = \beta$.

countries and across regions within countries. In other words, a lower level of human development is associated with a larger loss in human development for existing inequality.

The 2010 HDR has replaced the GDI with the Gender Inequality Index (GII). The GII is motivated by Seth (2009) and thus, unlike the GDI, it uses row-first aggregation. First, the achievements of each gender group are aggregated using a geometric mean ($\beta = 0$) to obtain an aggregate achievement. In the second stage, the aggregate achievements of both genders are aggregated using a harmonic mean ($\alpha = -1$) to obtain the level of gender inequality-adjusted human development. The normalised shortfall of this overall index from the level of human development with perfect gender equality is the GII.[33]

Decancq and Lugo (2012b) perform an empirical application on two families of multidimensional inequality indices based on Gini social evaluation functions: one family uses a column-first aggregation (Gajdos and Weymark 2005) and thus is insensitive to association; whereas another family uses a row-first aggregation (Decancq and Lugo, 2012b) and is thus sensitive to association. They use the Russian Longitudinal Monitoring Survey (RLMS-HSE) datasets, computing both indices for 1995 and 2005. They consider four dimensions: equivalent real expenditure, health, years of schooling and housing. For this application, however, the authors cardinalise some of the non-cardinal indicators. The indicators used for the health and the housing dimensions, such as self-assessed health status, access to various services, etc., are non-cardinal by nature. Decancq and Lugo find that the two families exhibit differences whenever indices within each family are made to be more sensitive to lower achievements.[34] Through a simulation exercise (from the observed data) that increases the association between dimensions, it is shown that the assessed levels and trends of inequality differ substantially when judged with association-sensitive indices as opposed to when judged with association-insensitive indices.

Using the same family of indices as Seth (2009), Decancq (2017) proposes and estimates a variant of the OECD's 'Better Life Index' (BLI), making it sensitive to inequality. While the BLI is a composite index and uses aggregated data, the proposed distribution-sensitive BLI (DS-BLI) requires micro-data and uses row-first aggregation. The indicators considered in the empirical exercise are similar to the original BLI, but with the required adjustments and using the Gallup World Poll survey. The considered variables are per capita income, employment, satisfaction with health and absence of health problems, years of schooling, social network support, satisfaction with water quality and air quality, self-reported safety and life satisfaction. Findings suggest that incorporating inequality does change country rankings. As in case

33 See the technical note *here*.

34 It refers to the value of the corresponding parameter δ in Table 1. The parameter assigns larger weights to lower achievements for Gajdos and Weymark (2005); whereas it assigns a larger weight to lower levels of human development for Decancq and Lugo (2012b).

of the HDI (Seth and Villar, 2017b), countries with a lower BLI ranking tend to have a larger loss of well-being due to multidimensional inequality.

Reviewing the applications of multidimensional inequality indices so far one may extract a few general observations. In the first place, the field seems to be still very fertile for further empirical investigations. Applications so far are relatively few and yet all of them reveal interesting insights about the effects of incorporating inequality into the measurement of human wellbeing or development. Second, some of the most prominent applications have used column-first aggregation, despite the recognised importance of considering the joint distribution of achievements. This may be no coincidence. Two factors may be at play. First, row-first aggregation types of measures require micro-data on the relevant variables used to measure inequality. This in itself may be not such a great limitation, as the availability of household survey micro-data keeps increasing. However, the variables needed to implement measures sensitive to the joint distribution need to be of a cardinal nature, or, otherwise, require a prior **cardinalisation** procedure (as performed in the work by Decancq and Lugo, 2012b and Decancq, 2017). While life expectancy can be considered a cardinal variable (although even here there might be disagreement), finding an equivalent meaningful cardinal variable at the individual level is not so straightforward. Thus, it may be that the combined requirements, namely cardinally meaningful variables at the micro-data level, do represent a practical limitation in the implementation of multidimensional inequality indices. Note, in fact, that the recent surge in implementations of sister measures of multidimensional poverty has been with measures that work with dichotomised achievements, facilitating the use of ordinal variables, which are predominant in multidimensional analysis.

6. Concluding Remarks

The technology for measuring multidimensional inequality has evolved enormously and has a lot of potential for monitoring human development. Considering what we have presented, we shall end with two final remarks.

In the first place, while there seems to be an increasing consensus regarding transfer properties (PBT has overcome controversies posed by UPD and UM), there is still debate over association-sensitive properties, which, in fact, reflects the discussion on whether dimensions of development are substitutes or complements.

Second, the framework of multidimensional inequality measurement is a rigorous technical framework. Yet it is the actual selection of dimensions and indicators of relevant capabilities and functionings – something briefly discussed in the introduction – that can make it operational for the measurement of human development. Real world data frequently pose significant limitations that require careful assessment and assumptions. This is no trivial matter, and, while it exceeds the scope of this paper, it is worth remembering. The fact that many of the typically available variables are of an ordinal nature, combined with the micro-data requirements when considering the joint distribution within the measures, poses an additional challenge, requiring further research.

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