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A Class of Association Sensitive Multidimensional Welfare Indices

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Abstract

The last few decades have seen increased theoretical and empirical interest in multidimensional measures of welfare. This paper develops a two-parameter class of welfare indices that is sensitive to two distinct forms of inter-personal inequality in the multidimensional framework. The first form of inequality pertains to the spread of each dimensional achievement across the population, as would be reflected in the multidimensional version of the usual Lorenz criterion. The second one regards association or correlation across dimensions, reflecting the key observation that inter-dimensional association may alter evaluation of individual as well as overall inequality. Most existing multi-dimensional welfare indices are, however, either completely insensitive to inter-personal inequality or are only sensitive to the first. The class of indices developed in this paper is sensitive to both forms of multidimensional inequality. An axiomatic characterization of the class is provided, and it is shown that other multidimensional indices, such as the ones developed by Bourguignon (1999) and Foster, Lopez-Calva, and Székely (2005), are sub-classes of this new broader class. Finally, essential statistical tests are constructed to verify the reliability of the evaluations generated by the indices.

Keywords: welfare measurement, inequality, distributive policy, inter-dimensional interaction, multidimensional association

JEL classification: O15, O2, D63, I31, I38, H53

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1 Introduction

Measurement of social welfare has always been a challenging task for economic theorists and policy makers across the globe. It is now universally agreed that economic affluence, often measured in terms of income, cannot be an exclusive indicator of social welfare, as it completely ignores the importance of various other aspects, such as education, and health. The advent of the basic needs approach and the capability approach primarily due to Amartya Sen has motivated the measurement of social welfare to be multidimensional in nature and has inspired a number of multidimensional indices on welfare, poverty, and inequality.

In this paper, a two-parameter class of multidimensional welfare indices is developed that is sensitive to the existing inequality across persons. The consideration of inter-personal inequality is essential during welfare evaluations because high inequality is detrimental to social welfare (Sen 1997). There are two distinct forms of inequality in the multidimensional context. The first form pertains to the spread of each dimensional achievements across the population (Kolm 1977); whereas, the second is related to correlation or more precisely - association among dimensions (Atkinson and Bourguignon 1982). The first form of inequality is called the *distribution sensitive inequality*; whereas, the second is called the *association sensitive inequality*. The class of generalized mean based welfare indices, developed in this paper, is sensitive to both these forms.

A range of multidimensional poverty and inequality indices have already been proposed over the past few decades incorporating both forms of inequality (Tsui 1995, Tsui 1999, Bourguignon 1999, Tsui 2002, Bourguignon and Chakravarty 2003, Decancq and Lugo 2008). However, most of the existing welfare indices are either completely insensitive to inter-personal inequality (the Human development index, various physical quality of life indices) or are only sensitive to distribution (Hicks 1997, Foster, Lopez-Calva, and Szekely 2005).

The class of indices, introduced in this paper, comprises of many nice features that are inherited from generalized means. This particular class is also characterized by a set of basic axioms. Apart from being sensitive to both forms of inequality, all indices in this class are subgroup consistent, which requires that increase in welfare of one group should lead to increase in the overall welfare, while that of the other group remained unaltered. This broader class includes few other existing classes of welfare indices, such as the ones developed by Bourguignon (1999) and Foster, Lopez-Calva, and Székely (2005). In addition to these various nice features, this class is amiable to empirical application due to its simple functional representation. To verify the reliability of the evaluations generated by these indices, appropriate statistical tests have also been developed.

The sections of this paper are organized as follows. In the second section, we discuss the notations that are followed in the rest of the paper. In the third section, the class of indices based on generalized means is introduced. The fourth section outlines the non-distributional axioms and provides a characterization of the functional form of the class introduced in the previous section. The fifth section introduces the association and the distribution sensitive axioms, and derives appropriate restrictions on parameters that enable the class to be sensitive to both forms of inequality. The sixth section is devoted towards the construction of statistical tests that are helpful in verifying the reliability of the evaluations generated by these indices. The final section discusses the scope for further research, various extensions of this study, and concludes this paper.

2 Notation

In this section, we introduce notations that are used throughout this paper. Let $\mathbb{R}^k$ denote the Euclidean k-space, and $\mathbb{R}^k_+, \mathbb{R}^k_{++} \subset \mathbb{R}^k$ denote the non-negative and the strictly positive k-spaces, respectively. Let $\mathbb{N}$ stand for the set of positive integers, $\mathbf{N} = \{1, \ldots, N\} \subset \mathbb{N}$ represents the set of N persons, and $\mathbf{D} = \{2, \ldots, D\} \subset \mathbb{N}$ is the set of fixed number of D dimensions. For every $M \in \mathbb{N}$ and for every $x, y \in \mathbb{R}^M$, we define $(x \vee y) = (\max(x_1, y_1), \ldots, \max(x_M, y_M))$ and $(x \wedge y) = (\min(x_1, y_1), \ldots, \min(x_M, y_M))$. For every $M \in \mathbb{N}$, any weight vector is denoted by $a \in \mathbb{R}^M_+$ such that $\Sigma_{m=1}^M a_m = 1$ and any equal weight vector is denoted by $\bar{a} \in \mathbb{R}^M_+$ such that $\bar{a}_m = 1/M \; \forall m = 1, \ldots M$. For every $M, r \in \mathbb{N}$ and for every $z \in \mathbb{R}^M$, $[z]_r$ is a replication vector where z is replicated r times such that $[z]_r = (z, \ldots, z) \in \mathbb{R}^{r \times M}$. Likewise, for every $Y \in \mathbb{R}^{LM}$, $[Y]_r$ is a replication matrix where Y is replicated r times such that $[Y]_r \in \mathbb{R}^{(r \times L)M}$. For every $L, M \in \mathbb{N}$, $\mathbf{1}_{LM} \in \mathbb{R}^{LM}$ is a matrix with every element equal to 1. Similarly, for every $M \in \mathbb{N}$, $\mathbf{1}_M \in \mathbb{R}^M$ is a vector with every element equal to 1.

For any $\mathbf{N} \subset \mathbb{N}$, an achievement[1] matrix is denoted by $H \in \mathbb{R}^{ND}_{++}$ and the set of all such matrices by $\mathcal{H} = \cup_{\mathbf{N} \subset \mathbb{N}} \mathbb{R}^{ND}_{++}$. An achievement matrix with a fixed number of population $N \in \mathbb{N}$ is denoted by H_N and the set of all such matrices by $\mathcal{H}_N = \cup \mathbb{R}^{ND}_{++}$. Let h_{nd}, the nd^{th} element in H, be the achievement of person n in dimension $d \; \forall n \in \mathbf{N}$ and $\forall \; d \in \mathbf{D}$. Row n and column d in H are denoted by $h_{n\cdot} \; \forall n \in \mathbf{N}$ and $h_{\cdot d} \; \forall \; d \in \mathbf{D}$, respectively. A social welfare index is defined by $W : H \to \mathbb{R}$. A society $\mathcal{A}$ has weakly (strictly) higher social welfare than another society $\mathcal{B}$ if and only if $W\left(H^{\mathcal{A}}\right) \geq (>) \; W\left(H^{\mathcal{B}}\right)$ for any $H^{\mathcal{A}}, H^{\mathcal{B}} \in \mathcal{H}$.

3 A Class of Indices

The class of social welfare indices that is developed in this paper is based on generalized means, which can be defined as follows[2]. For every $M \in \mathbb{N}$, for every $x \in \mathbb{R}^M_{++}$, for every $a \in \mathbb{R}^M_+$, and for every $\gamma \in \mathbb{R}$, the generalized mean of order γ is defined by:

$$\mu_\gamma(x; a) = \begin{cases} \left(\sum_{m=1}^M a_m x_m^\gamma\right)^{1/\gamma} & \text{for } \gamma \neq 0 \\ \prod_{m=1}^M x_m^{a_m} & \text{for } \gamma = 0 \end{cases}.$$

For $\gamma = 1$, the generalized mean is reduced to the weighted arithmetic mean. It is equivalent to the weighted geometric mean and the weighted harmonic mean for $\gamma = 0$ and $\gamma = -1$, respectively.

A particular class of generalized means is the one where all elements under consideration receive equal weight and is defined as follows. For every $M \in \mathbb{N}$, for every $x \in \mathbb{R}^M_{++}$, for $\bar{a} \in \mathbb{R}^M_+$, and for every $\gamma \in \mathbb{R}$, the equal weighted version of generalized means of order

[1] We begin with the assumption that achievements are normalized in some way or the other.

[2] The properties of generalized mean can be found in Appendix B.

γ is defined by:

$$\mu_\gamma(x;\bar{a}) = \begin{cases} \left(\frac{1}{M}\sum_{m=1}^{M} x_m^\gamma\right)^{1/\gamma} & \text{for } \gamma \neq 0 \\ \left(\prod_{m=1}^{M} x_m\right)^{1/M} & \text{for } \gamma = 0 \end{cases}.$$

We denote the simple arithmetic mean for any $x \in \mathbb{R}_{++}^M$ by $\mu(x;\bar{a})$; whereas, the weighted arithmetic mean is denoted by $\mu(x;a)$.

The multidimensional social welfare index is constructed from any achievement matrix in two steps. In the first step, the standardized achievement of each person is calculated by aggregating the achievements in all D dimensions. In the second step, the social welfare index is obtained by aggregating the standardized achievements of all persons. The standardized achievements are aggregated by function $Q : \mathbb{R}_{++}^D \to \mathbb{R}_{++}$, where Q is identical across persons and is called the *individual aggregation function* (IAF). Likewise, the standardized achievements are aggregated by function $\Phi : \mathbb{R}_{++}^N \to \mathbb{R}$, where Φ is called the *standardized achievement aggregation function* (SAAF).

An important objective while constructing the class of indices should be to enhance its empirical applicability by making it easily comprehensible and analytically transparent. The usual practice during aggregation is to attach weights to each dimension to highlight the relative importance of the concerned dimensions. Moreover, during the first step aggregation, it is often a matter of interest for the policy makers to understand how various dimensions contribute to the standardized achievement of each person. These issues can be easily dealt with if the IAF is assumed to be additively separable. Then, for every $n \in \mathbf{N}$ and for every $h_{n\cdot} \in \mathbb{R}_{++}^D$, the IAF can be expressed as:

$$Q(h_{n\cdot}) = U(V_1(h_{n1}) + \ldots + V_D(h_{nD})); \tag{1}$$

where U is continuous and V_d is continuous for all $d = 1, \ldots, D$. Therefore, for every $\mathbf{N} \subset \mathbb{N}$ and for every $H_N \in \mathcal{H}$, the multidimensional social welfare index is defined as:

$$W(H_N) = \Phi\left(U\left(\sum_{d=1}^{D} V_d(h_{1d})\right), \ldots, U\left(\sum_{d=1}^{D} V_d(h_{Nd})\right)\right). \tag{2}$$

In this paper, the following two-parameter class of social welfare indices is proposed. The class is based on generalized means. For every $\mathbf{N} \subset \mathbb{N}$, for every $H_N \in \mathcal{H}$, for every $\alpha, \beta \in \mathbb{R}$, for every $a \in \mathbb{R}_+^D$, and for $\bar{a} \in \mathbb{R}_+^N$, the class of social welfare indices is defined as:

$$\mathcal{W}(H_N;\alpha,\beta,a,\bar{a}) = \mu_\alpha\left(\mu_\beta(h_{1\cdot};a), \ldots, \mu_\beta(h_{N\cdot};a);\bar{a}\right). \tag{3}$$

It is shown in the next section that the class of social welfare indices given by (3) is the natural class, which can be obtained from (2) based on the following set of basic axioms.

4 Non-Distributional Axioms

In this section, we introduce the non-distributional axioms that enables a class of welfare indices to be easily presentable and technically sound, at the same time.

The first axiom prevents the level of social welfare to change abruptly due to a change in the achievement of any person in any dimension.

Continuity (CNT)**.** For every $\mathbf{N} \subset \mathbb{N}$ and for every $H_N \in \mathcal{H}$, $W(H_N)$ is continuous on $\mathbb{R}^{ND}_{++}$.

The next two axioms make the interpretation of the welfare indices easy and attractive. According to the first of them, if a person has equal achievement in all dimensions then there is no harm to assume that the standardized achievement is also equal to any of the achievements. Moreover, if all persons have the same level of standardized achievements then we assume the social welfare index to be equal to that standardized achievement.

Normalization (NM)**.** For every $\mathbf{N} \subset \mathbb{N}$, for every $\zeta > 0$, and for every $H_N \in \mathcal{H}$ such that $H_N = \zeta \mathbf{1}_{ND}$,

$$Q(h_{n\cdot}) = \zeta \ \forall n \in \mathbf{N} \text{ and } W(H_N) = \zeta.$$

Secondly, we assume that the preference is *homothetic* as it is easy to work with, and linear homogeneity, a special case of homothetic preference, makes the social welfare evaluation easily comprehensible. Thus, according to the second of these two axioms, if all achievements are changed proportionally, the social welfare also changes by the same proportion.

Linear Homogeneity (LH)**.** For every $\mathbf{N} \subset \mathbb{N}$, for every $\delta > 0$, and for every $H_N, H'_N \in \mathcal{H}_N$ such that $H'_N = \delta H_N$,

$$W(H'_N) = \delta W(H_N).$$

While measuring social welfare, identity of a person should not ethically receive any significance. The next axiom ensures that we treat all persons as being anonymous and with equal importance.

Symmetry in People (SP)**.** For every $\mathbf{N} \subset \mathbb{N}$, for every $H_N, H'_N \in \mathcal{H}_N$, and for every permutation matrix[3] $P \in \mathbb{R}^{NN}_{+}$ such that $H'_N = PH_N$,

$$W(H'_N) = W(H_N).$$

None of the three axioms so far allows the population of a society vary. As we often perform cross-societal comparisons, we require an axiom that allows us to compare societies with varying population size. The axiom of *population replication invariance* guarantees that if the population of a society is replicated several times with the respective achievement vectors unaltered, then the level of social welfare remains unchanged.

Population Replication Invariance (PRI)**.** For every $r \in \mathbb{N}$ and for every $H, H' \in \mathcal{H}$ such that $H' = [H]_r$,

$$W(H') = W(H).$$

The next axiom is called *monotonicity*. This axiom requires that if the achievement of a person in a dimension increases, while that of the rest unaltered, the social welfare should strictly increase. This axiom also requires that the standardized achievement of a person increases owing to an increase in any of the achievements.

[3] A permutation matrix is a square matrix with each row and column have exactly one element equal to one and rest equal to zero. An identity matrix is a special type of permutation matrix.

Monotonicity (MO)**.** (*i*) For every $\mathbf{N} \subset \mathbb{N}$ and for every $H_N, H'_N \in \mathcal{H}_N$ such that $H'_N \geq H_N$ and $H'_N \neq H_N$,

$$W\left(H'_N\right) > W\left(H_N\right).$$

(*ii*) For every $n \in \mathbf{N}$, for every $h_n, h'_n \in \mathbb{R}^D_{++}$ such that $h'_n \geq h_n$ and $h'_n \neq h_n$,

$$Q\left(h'_n\right) > Q\left(h_n\right).$$

This axiom implicitly assumes that no personal achievement is harmful for a society. This axiom is, however, silent if the welfare of an entire group of people changes. An improvement in welfare of a group of people can be accompanied both by improvement in achievements of some people while deterioration in achievements for others, at the same time. The social welfare for the entire society is required to increase if the welfare of a group increases, while that of the rest unaltered.

Subgroup Consistency (SC)**.** For every $N_1, N_2, N \in \mathbb{N}$ such that $N_1 + N_2 = N$, for every $H_{N_1}, H'_{N_1} \in \mathcal{H}_{N_1}$, and for every $H_{N_2}, H'_{N_2} \in \mathcal{H}_{N_2}$, if $W\left(H'_{N_1}\right) > W\left(H_{N_1}\right)$ and $W\left(H'_{N_2}\right) = W\left(H_{N_2}\right)$, then $W\left(H'_{N_1}, H'_{N_2}\right) > W\left(H_{N_1}, H_{N_2}\right)$.

Based on the set of non-distributional axioms, we characterize the class of social welfare indices in (3) by Theorem 1. We show that the functional form of the social welfare indices in (3) is both necessary and sufficient for the social welfare functions of the form in (2) if all non-distributional axioms are satisfied.

Theorem 1 *For every* $\mathbf{N} \in \mathbb{N}$ *and for every* $H \in \mathcal{H}$*, a social welfare index of the form in* (2) *satisfies CNT, NM, LH, SP, PRI, MO, and SC if and only if it is of the form:*

$$\mathcal{W}\left(H; \alpha, \beta, a, \bar{a}\right) = \mu_\alpha\left(\mu_\beta\left(h_{1\cdot}; a\right), \ldots, \mu_\beta\left(h_{N\cdot}; a\right); \bar{a}\right)$$

for all $\alpha, \beta \in \mathbb{R}$*, for every* $a \in \mathbb{R}^D_+$*, and* $\bar{a} \in \mathbb{R}^N_+$*.*

Proof. See Appendix A. ■

Thus, we derive the functional form of the social welfare index in Theorem 1 that satisfies all of the non-distributional axioms introduced in this section. However, none of these axioms enabled the class of indices to be sensitive to the existing inter-personal inequality. In other words, the non-distributional axioms leave the indices to be insensitive to the distribution of achievements across the population. In the next section, we introduce the distributional axioms and set appropriate restrictions on the parameters that allow the class satisfy these axioms.

5 Distributional Axioms

There are two distinct forms of inequality in the multidimensional context. One is the *distribution sensitive inequality* (Kolm 1977) and the other is the *association sensitive inequality* (Atkinson and Bourguignon 1982). In this section, we discuss both these forms of multidimensional inequality in detail and introduce the corresponding axioms. We derive proper restrictions on parameters α and β that allow the indices to be sensitive to both form of multidimensional inequality.

5.1 Association Sensitive Inequality and Axioms

That inter-dimensional association is important has already been emphasized repeatedly in the previous studies (Tsui 1995, Tsui 1999, Bourguignon 1999, Tsui 2002, Bourguignon and Chakravarty 2003). The role of inter-dimensional correlation in the study of welfare analysis has been introduced by Atkinson et. al. (1982), where correlation between two dimensions can be increased, leaving the marginal distribution of dimensions unaltered. However, an achievement matrix can be obtained from another one by increasing correlation among dimensions in various ways. For example, Bourguignon et. al. (2003) introduced a concept called *Correlation Increasing Switch* (CIS); whereas, Tsui (1999) defined a concept called *Correlation Increasing Transfer*[4] (CIT) following the notion of *basic rearrangement* due to Boland and Proschan (1988). We pursue the later approach and call it *association increasing transfer*[5]. The formal definition of the concept is provided next.

For every $\mathbf{N} \subset \mathbb{N}$ and $N > 1$, and for every $H'_N, H_N \in \mathcal{H}_N$, H'_N is obtained from H_N by an association increasing transfer if $H'_N \neq H_N$, H'_N is not a permutation of H_N, and there exist two persons n_1 and n_2 such that $h'_{n_1\cdot} = (h_{n_1\cdot} \vee h_{n_2\cdot})$, $h'_{n_2\cdot} = (h_{n_1\cdot} \wedge h_{n_2\cdot})$, and $h'_{n\cdot} = h_{n\cdot}\ \forall\ n \neq n_1, n_2$. Intuitively, association among dimensions increases if one person has strictly higher achievement in some dimensions but strictly lower in others before transfer, and obtains higher achievement in all dimensions than the other does after the transfer takes place[6]. Based on the concept of association increasing transfer, the following set of axioms are develop .

Strictly Decreasing under Increasing Association (*SDIA*)**.** For every $\mathbf{N} \subset \mathbb{N}$ and $N > 1$, and for every $H'_N, H_N \in \mathcal{H}_N$ such that H'_N is obtained from H_N by a finite sequence of association increasing transfers,

$$W\left(H'_N\right) < W\left(H_N\right).$$

Strictly Increasing under Increasing Association (*SIIA*)**.** For every $N \in \mathbb{N}$ and $N > 1$, and for every $H'_N, H_N \in \mathcal{H}_N$ such that H'_N is obtained from H_N by a finite sequence of association increasing transfers,

$$W\left(H'_N\right) > W\left(H_N\right).$$

The corresponding weak versions of the axioms are *weakly decreasing under increasing association* (WDIA) and *weakly increasing under increasing association* (WIIA), which include equalities along with the strict inequalities. The next obvious question is what restrictions on parameters α and β enable the class of welfare indices in (3) to be strictly sensitive to association. The following theorem summarizes the desired restrictions.

[4] The definition of CIT and CIS are equivalent if there are only two dimensions. However, if there are more than two dimensions, the definition of CIS is bit confusing as it requires increasing the correlation between any two dimensions, leaving the correlation between the rest unaltered.

[5] The term 'association' rather than the term 'correlation' is preffered since the term 'association' is much broader than the term 'correlation'. In statistical literature, correlation simply means Pearson's product moment correlation.

[6] The definition of CIS is different in the sense that it considers increasing correlation between two dimensions only.

Theorem 2 *For every* $\mathbf{N} \subset \mathbb{N}$ *and* $N > 1$*, for every* $H \in \mathcal{H}$*, for every* $a \in \mathbb{R}^D_+$*, for* $\bar{a} \in \mathbb{R}^N_+$*, and for every* $\alpha, \beta \in \mathbb{R}$*, (i)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies SDIA if and only if* $\alpha < \beta$*. (ii)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies SIIA if and only if* $\alpha > \beta$*.*

Proof. See Appendix B. ■

Thus, Theorem 2 imposes some restrictions on the parameters α and β so that the proposed class of welfare indices satisfies *SDIA* and *SIIA*.[7]

5.2 Distribution Sensitive Inequality and Axioms

The other form of multidimensional inequality is the distribution sensitive inequality, which is based on the concept of distribution sensitivity in the single dimensional context (where $D = 1$). A unidimensional distribution is stated to be more equal than another distribution, if the former is obtained from the latter by a finite sequence of Pigou-Dalton transfers. Distribution $x = (x_1, \ldots, x_N)$ is obtained from another distribution $y = (y_1, \ldots, y_N)$ by a Pigou-Dalton transfer if $x \neq y$, $x_{n_1} = \lambda y_{n_1} + (1 - \lambda) y_{n_2}$, $x_{n_2} = (1 - \lambda) y_{n_1} + \lambda y_{n_2}$, and $x_n = y_n \; \forall n \neq n_1, n_2$, where $\lambda \in (0, 1)$. According to an alternative definition, a distribution x is obtained from another distribution y by a finite sequence of Pigou-Dalton transfers if and only if $x = By$, where B is a bistochastic matrix[8]. The concept of *common smoothing* builds on the multidimensional extension of this later construct. For every $\mathbf{N} \subset \mathbb{N}$ and $N > 1$, and for every $H'_N, H_N \in \mathcal{H}_N$, H'_N is obtained from H_N by common smoothing if there exists a bistochastic matrix B, which is not a permutation matrix, such that $H'_N = BH_N$. Note that a finite sequence of Pigou-Dalton transfers in the multidimensional context is not equivalent to common smoothing. This equivalence relation holds for $N = 2$ or $D = 1$, but does not hold both ways for $N \geq 3$ and $D \geq 2$. The following two axioms relate the notion of common smoothing to distribution sensitive multidimensional social welfare indices.

Strictly Increasing under Common smoothing (SICS)**.** For every $\mathbf{N} \subset \mathbb{N}$ and $N > 1$, for every $H'_N, H_N \in \mathcal{H}_N$, and for every bistochastic matrix $B \in \mathbb{R}^{NN}_+$ such that $H'_N = BH_N$,

$$W(H'_N) > W(H_N).$$

The weak version of this axiom is *weakly increasing under common smoothing* (WICS) that includes the case $W(H'_N) = W(H_N)$ apart from the strict inequality. Now, we need to derive the set of restrictions that allows $\mathcal{W}(\cdot)$ to satisfy the axioms of distribution sensitivity. Theorem 3 summarizes the restrictions on parameters α and β.

Theorem 3 *For every* $N \in \mathbb{N}$ *and* $N > 1$*, for every* $H_N \in \mathcal{H}_N$*, for every* $a \in \mathbb{R}^D_+$*, and for* $\bar{a} \in \mathbb{R}^N_+$*, (i)* $\mathcal{W}(H_N; \alpha, \beta, a, \bar{a})$ *satisfies SICS if and only if* $\alpha, \beta \leq 1$ *and* $\alpha = \beta \neq 1$*, (ii)* $\mathcal{W}(H_N; \alpha, \beta, a, \bar{a})$ *satisfies WICS if and only if* $\alpha, \beta \leq 1$*.*

Proof. See Appendix C. ■

In the following section, we combine all the restrictions provided by the above theorems and present the class of indices that satisfies the desired axioms.

[7] The corresponding results containing WDIA and WIIA can be easity obtained. It can be shown that $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ satisfies WDIA (WIIA) if and only if $\alpha \leq \beta$ ($\alpha \geq \beta$).

[8] A bistochastic matrix is a non-negative square matrix whose row sum and column sum are both equal to one.

6 The Class of Association Sensitive Welfare Indices

In the previous sections, we have obtained various restrictions on α and β that if imposed would enables $\mathcal{W}(\cdot)$ to satisfy the desired axioms. According to Theorem 1, the class satisfies the set of non-distributional axioms for all $(\alpha, \beta) \in \mathbb{R}^2$. Theorem 2 states that the class is strictly sensitive to association for $\alpha \neq \beta$. Finally, the class is strictly sensitive to common smoothing for $\alpha, \beta \leq 1$ and $\alpha = \beta \neq 1$.

We combine these three theorems to define the following class of indices that is strictly sensitive to both forms of inequality.

Theorem 4 *For every* $\mathbf{N} \subset \mathbb{N}$ *and* $N > 1$*, for every* $H \in \mathcal{H}$*, for every* $a \in \mathbb{R}_+^D$*, and for* $\bar{a} \in \mathbb{R}_+^N$*, (i)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies CNT, NM, LH, SP, PRI, MO, SC, SICS, and SDIA if and only if* $\alpha < \beta \leq 1$*. (ii)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies CNT, NM, LH, SP, PRI, MO, SC, SICS, and SIIA if and only if* $\beta < \alpha \leq 1$*.*

Proof. The proof is straight forward and directly follows from Theorem 1, Theorem 2, and Theorem 3. ■

Theorem 4 shows that the proposed class of indices is strictly sensitive to both forms of inequality if and only if $\alpha, \beta \leq 1$ and $\alpha \neq \beta$. However, a careful analysis would reveal that the proposed class includes various other existing class of welfare indices.

If we set $\alpha = \beta = 1$, we have the following welfare index:

$$\mathcal{W}(H; 1, 1, a, \bar{a}) = \mu\left(\mu\left(h_{1\cdot}; a\right), \ldots, \mu\left(h_{N\cdot}; a\right), \bar{a}\right). \tag{4}$$

The social welfare in (4) is a simple mean of simple weighted means and $\alpha = \beta = 1$ implies that the index is not strictly sensitive at all to any form of multidimensional inequality. A further manipulation would lead to the following form:

$$\mathcal{W}(H; 1, 1, a, \bar{a}) = \mu\left(\mu\left(h_{\cdot 1}; \bar{a}\right), \ldots, \mu\left(h_{\cdot D}; \bar{a}\right), a\right). \tag{5}$$

Form (5) is highly familiar as this is applied while calculating well-known welfare indices such as the human development index (UNDP 2006) and a range of physical quality of life indices (Morris 1979).

Next, a restriction of $\beta = \alpha < 1$ leads to the following class of welfare indices:

$$\mathcal{W}(H; \alpha, \alpha, a, \bar{a}) = \mu_\alpha\left(\mu_\alpha\left(h_{1\cdot}; a\right), .., \mu_\alpha\left(h_{N\cdot}; a\right), \bar{a}\right). \tag{6}$$

The class in (6) is strictly distribution sensitive but is not strictly association sensitive. Again, a quick manipulation of (6) would give us another familiar class of indices:

$$\mathcal{W}(H; \alpha, \alpha, a, \bar{a}) = \mu_\alpha\left(\mu_\alpha\left(h_{\cdot 1}; \bar{a}\right), .., \mu_\alpha\left(h_{\cdot D}; \bar{a}\right), a\right). \tag{7}$$

This class of welfare indices is proposed by Foster, Lopez-Calva, and Székely (2005).

Note that the only difference between both functional forms in (6) and (7) is that the sequence of aggregation has been altered. Instead of first aggregating across dimensions, if the aggregation takes place first across people, the resulting welfare evaluation does not change. The axiom that leads the class of indices to be invariant to the order of aggregation is called *path independence* due to Foster et. al. (2005).

Path Independence (PI)**.** For every $N \subset \mathbf{N}$ and for every $H_N \in \mathcal{H}_N$,

$$\Phi\left(Q\left(h_{1\cdot}\right),\ldots,Q\left(h_{N\cdot}\right)\right)=Q\left(\Phi\left(h_{\cdot 1}\right),\ldots,\Phi\left(h_{\cdot D}\right)\right).$$

According to this axiom, the sequence of aggregation does not matter. The axiom is especially important under the circumstances when the data for different dimensions are available at different disaggregated levels. For example, education data may be available at the individual level, income data may be available at the household level, health data may be available at the municipality level etc. In this situation, we might not have enough information to infer association among dimensions. The following Theorem defines the class of indices that satisfies path independence beside satisfying other desirable axioms.[9]

Theorem 5 *For every $\mathbf{N} \subset \mathbb{N}$ and $N > 1$, for every $H \in \mathcal{H}$, for every $a \in \mathbb{R}_+^D$, and for $\bar{a} \in \mathbb{R}_+^N$, (i) $\mathcal{W}(H;\alpha,\beta,a,\bar{a})$ satisfies CNT, NM, LH, SP, PRI, MO, SC, and PI if and only if $\alpha=\beta$, (ii) $\mathcal{W}(H;\alpha,\beta,a,\bar{a})$ satisfies CNT, NM, LH, SP, PRI, MO, SC, SICS, and PI if and only if $\beta=\alpha<1$.*

Proof. By Theorem 1, we know that $\mathcal{W}(H;\alpha,\beta,a,\bar{a})$ satisfies all non-distributional axioms. Define $W_1=\mu_\alpha\left(\mu_\beta\left(h_{1\cdot};a\right),\ldots,\mu_\beta\left(h_{N\cdot};a\right);\bar{a}\right)$ and $W_2=\mu_\beta\left(\mu_\alpha\left(h_{\cdot 1};\bar{a}\right),..,\mu_\alpha\left(h_{\cdot D};\bar{a}\right),a\right)$. It is straight forward to show that if $\alpha=\beta$ then $W_1=W_2$. Next, we show $W_1\neq W_2$ if $\alpha\neq\beta$. According to Theorem 26 of Hardy, Littlewood, and Pólya (1934), $W_1>W_2$ for $\beta<\alpha$, and $W_1<W_2$ for $\beta>\alpha$.[10]

The second part of the theorem directly follows by applying Theorem 3 in addition. ∎

However, it is evident from Theorem 5 that the proposed class of indices can not be strictly sensitive to association if it satisfies path independence because path independence requires $\alpha=\beta$.

The class of strictly association sensitive indices contains another existing class of welfare indices that is proposed by Bourguignon (1999), while commenting on the class of inequality indices proposed by Maasoumi (1999). For every $N \subset \mathbf{N}$, for every $H_N \in \mathcal{H}_N$, for every $a \in \mathbb{R}_+^D$, for $\bar{a} \in \mathbb{R}_+^N$, for every $0<\alpha<1$, and for every $\beta<1$, the Bourguignon class of welfare indices can be defined as:

$$W_B\left(H;\alpha,\beta,a,\bar{a}\right)=\left(\mu_\alpha\left(\mu_\beta\left(h_{1\cdot};a\right),\ldots,\mu_\beta\left(h_{N\cdot};a\right);\bar{a}\right)\right)^\alpha=\left(\mathcal{W}\left(H;\alpha,\beta,a,\bar{a}\right)\right)^\alpha.$$

Therefore, for $\beta<1$ and $0<\alpha<1$, $\mathcal{W}(H;\alpha,\beta,a,\bar{a})$ is simply a monotonic transformation of the Bourguignon class of indices. This class of indices satisfies all non-distributional axioms and is sensitive to both forms of multidimensional inequality in strict sense whenever $\alpha\neq\beta$. However, the role of the inequality aversion parameter (α) is not transparent for two reasons. First of all, suppose that there are two societies with identical achievement vectors, with perfect equality across persons, and with identical values of parameter β. The only

[9] For a class of *path independent* standard of living index, see Dutta, Pattanaik, and Xu (2003).

[10] Although Hardy, Littlewood, and Pólya proves the theorem for $\alpha,\beta>0$, it can be easily extended for all $\alpha,\beta\in\mathbb{R}$.

difference is societies' aversions towards inequality. Bourguignon Index would yield different levels of social welfare for both societies. It is not apparent what causes this difference because inequality aversion should not play any role as there exist no inequality! In addition, it is not clear what value of the inequality aversion parameter leads to higher degree of inequality aversion. The degree of inequality aversion does not monotonically depend on the inequality aversion parameter.

7 Policy Recommendation

Once we have derived the class of association sensitive indices, let us check how this property influences the policy recommendation. Suppose, a policy maker is required to decide how to allocate her budget across the population. The obvious concern would be - *which person in the society should receive the first marginal assistance so that the welfare of the entire society can be maximized*. We devote this section to show how sensitivity to association among dimensions affect the decision of the policy maker. The class of social welfare function in (3) is based on generalized means and, thus, is differentiable.

If the policy maker provides the marginal assistance (\$) to person n to improve her achievement in dimension d, the increment in total welfare would be:

$$\frac{\partial \mathcal{W}\left(H;\alpha,\beta,a,\bar{a}\right)}{\partial \$} = \left(a_d h_{nd}^{\beta-1} C_n^{\alpha-\beta}\mathbf{C}\right) c_{nd};$$

where a_d is the share of dimension d in the standardized achievement of each person, $c_{nd} = \partial h_{nd}/\partial \$$ is the improvement in achievement h_{nd} due to the assistance, $C_n = \mu_\beta\left(h_{n\cdot};a\right)$ is the standardized achievement of person n, and $\mathbf{C} = \left(\mathcal{W}\left(H;\alpha,\beta,a,\bar{a}\right)\right)^{1-\alpha}$. The policy maker would assist person n to improve her achievement in dimension d if

$$\omega_{nd} > \omega_{\bar{n}\bar{d}}\ \forall \bar{n} \in \{1,\ldots,N\}/\{n\} \text{ and } \forall \bar{d} \in \{1,\ldots,D\}/\{d\}; \tag{8}$$

where, $\omega_{nd} = a_d h_{nd}^{\beta-1} C_n^{\alpha-\beta} c_{nd}$. Note that $\mathbf{C}$ is identical across all persons.

Let us provide some intuitive interpretation of condition (8). To simplify our analysis, we make the following two simplifying assumptions for a while. We assume that $a_d = a_{\bar{d}}$ $\forall d \neq \bar{d}$, i.e. the share of all dimensions in social welfare is the same; and $s_{nd} = s_{\bar{n}\bar{d}}$ $\forall d \neq \bar{d}, n \neq \bar{n}$, i.e. improvement in all dimensions for all persons from the dollar is the same. Let us consider the following circumstances.

First, suppose the policy maker has already decided to assist person n based on condition (8). As there are more than one dimension, the concern is which dimension of that person she should focus on. She would spend the dollar on dimension d if $\omega_{nd} = \max\left(\omega_{n\cdot}\right)$. A further manipulation of the condition would require that $h_{nd} = \min\left(h_{n\cdot}\right)$, i.e. she should provide the assistance to person n to improve the dimension with the lowest achievement.

In the second situation, suppose there are two persons n and $\bar{n}$ such that $\min\left(h_{n\cdot}\right) = \min\left(h_{\bar{n}\cdot}\right)$ but $C_n \neq C_{\bar{n}}$. Policy maker's decision is based on the relation between dimensions. If dimensions are substitutes, then higher correlation is detrimental to social welfare and $\alpha < \beta$. We already know that any person, if assisted, should receive the assistance to improve the dimension where his achievement is the lowest. Therefore, person n should receive the assistance instead of person $\bar{n}$ if $C_n^{\alpha-\beta} > C_{\bar{n}}^{\alpha-\beta}$, or, $C_n < C_{\bar{n}}$. On the other hand, complementarity requires $\alpha > \beta$ and person n should receive the assistance instead of person $\bar{n}$ if $C_n > C_n$.

In the third situation, suppose both $C_n \neq C_{\bar{n}}$ and $\min(h_{n\cdot}) \neq \min(h_{\bar{n}\cdot})$. Person n would receive the dollar if $(\min(h_{n\cdot}))^{\beta-1} C_n^{\alpha-\beta} > (\min(h_{\bar{n}\cdot}))^{\beta-1} C_{\bar{n}}^{\alpha-\beta}$. The decision does not only depend on the minimum achievement but also on the overall achievement. However, if the welfare index satisfies axiom PI then $\alpha = \beta$ is a requirement and person n would receive the assistance if $\min(h_{n\cdot}) < \min(h_{\bar{n}\cdot})$ only. The policy recommendation remains unaltered even if C_n is reasonably larger than $C_{\bar{n}}$, which should not always be a desirable conclusion for any distributive policy.

Without the simplifying assumptions, the general condition for assisting person n to improve dimension d is given by condition (8).

8 Statistical Tests

As we calculate the welfare index from a particular sample dataset, one might always question the reliability of the estimates and that motivates us to construct the statistical tests for the entire class of indices. For every $\mathbf{N} \subset \mathbb{N}$, for every $H_N \in \mathcal{H}_N$, for every $(\alpha, \beta) \in [-\infty, 1]$, for every $a \in \mathbb{R}_+^D$, and for $\bar{a} \in \mathbb{R}_+^N$, the estimate of the welfare index is:

$$\hat{\theta} = \mu_\alpha\left(\mu_\beta(h_{1\cdot}; a), \ldots, \mu_\beta(h_{N\cdot}; a); \bar{a}\right).$$

Thus, our dataset contains N individual specific observations on D dimensions. The estimate $\hat{\theta}$ can also be expressed as[11]:

$$\hat{\theta} = \left(\frac{1}{N}\sum_{n=1}^{N}\left(\sum_{d=1}^{D} a_d h_{nd}^{\beta}\right)^{\beta/\alpha}\right)^{1/\alpha} = \left(\frac{1}{N}\sum_{n=1}^{N}\eta_n\right)^{1/\alpha};$$

where $\eta_n = \left(\sum_{d=1}^{D} a_d h_{nd}^{\beta}\right)^{\beta/\alpha}$ $\forall n \in \mathbf{N}$. Let us define $\eta = (\eta_1, \ldots, \eta_N)$ and $\hat{\gamma} = \frac{1}{N}\sum_{n=1}^{N}\eta_n$.

We assume each observation is independently drawn from identical distributions. In other words, $\{h_{n1}, \ldots, h_{nD}\}_{n=1}^N$ are independently and identically distributed (*iid*). Consequently, $\{\eta_n\}_{n=1}^N$ are also *iid* since η_n is just a monotonic transformation of $h_{n\cdot}$ for all $n \in \mathbf{N}$. If $\{h_{n\cdot}\}_{n=1}^N$ are *iid* from multivariate distribution function $F(h_1, \ldots, h_D)$, the welfare index can be written as:

$$\theta = \left(\int \cdots \int \left(\sum_{d=1}^{D} a_d h_d^{\beta}\right)^{\beta/\alpha} dF(h_1, \ldots, h_D)\right)^{1/\alpha}.$$

To find a consistent estimator of $\hat{\theta}$, we first need to find a consistent estimator for $\hat{\gamma}$. As the sample observations are *iid*, we use the weak law of large number (WLLN) and have:

$$\hat{\gamma} \xrightarrow{p} E(\eta).$$

[11] The corresponding forms for $\alpha = 0$ or $\beta = 0$ can be calculated accordingly.

By continuous mapping theorem (CMT),

$$\hat{\theta} \xrightarrow{p} (E(\eta))^{1/\alpha}.$$

According to the central limit theorem (CLT), as $N \to \infty$, the distribution of $\hat{\gamma}$ converges to the normal distribution with mean γ and variance σ_γ^2. Thus,

$$\sqrt{N}(\hat{\gamma} - \gamma) \xrightarrow{D} \text{Normal}\left(0, \sigma_\gamma^2\right).$$

Using the delta method, we can have:

$$\sqrt{N}\left(\hat{\theta} - \theta\right) = \sqrt{N}\left(\hat{\gamma}^{1/\alpha} - \gamma^{1/\alpha}\right) \xrightarrow{D} \frac{1}{\alpha}\gamma^{\frac{1}{\alpha}-1} \text{Normal}\left(0, \sigma_\gamma^2\right).$$

Using $\gamma = \theta^\alpha$, we get

$$\sqrt{N}\left(\hat{\theta} - \theta\right) \xrightarrow{D} \frac{1}{\alpha}\theta^{1-\alpha} \text{Normal}\left(0, \sigma_\gamma^2\right) = \text{Normal}\left(0, \left(\theta^{1-\alpha}\sigma_\gamma^2\right)/\alpha\right).$$

Now, we estimate the standard error of estimate $\hat{\theta}$. We can estimate the variance of $\hat{\gamma}$ from the sample as:

$$\hat{\sigma}_\gamma^2 = \widehat{\text{var}}(\eta) = \frac{1}{N-1}\sum\nolimits_{n=1}^{N} (\eta_i - \bar{\eta})^2;$$

where $\bar{\eta} = \frac{1}{N}\sum_{n=1}^{n} \eta_n$. Therefore,

$$\hat{\sigma}_\theta^2 = \frac{1}{\alpha^2}\hat{\theta}^{2(1-\alpha)}\hat{\sigma}_\gamma^2 = \frac{\hat{\theta}^{2(1-\alpha)}}{\alpha^2(N-1)}\sum\nolimits_{n=1}^{N} (\eta_i - \bar{\eta})^2.$$

Hence, the standard error of the estimate is

$$SE\left(\hat{\theta}\right) = \frac{\hat{\sigma}_\theta}{\sqrt{N}} = \hat{\theta}^{(1-\alpha)}\sqrt{\frac{\sum_{n=1}^{N} (\eta_i - \bar{\eta})^2}{\alpha^2 N(N-1)}}.$$

Next, we calculate the confidence interval for the statistic. In this situation, both the mean and the variance are unknown. Thus, the test statistic is equal to

$$T = \frac{\hat{\theta} - \theta}{SE\left(\hat{\theta}\right)} = \frac{\sqrt{N}\left(\hat{\theta} - \theta\right)}{\hat{\sigma}_\theta}.$$

As we know that if $\sqrt{N}\left(\hat{\theta} - \theta\right) \xrightarrow{D} \text{Normal}(0, \sigma_\theta^2)$ and $\hat{\sigma}_\theta^2 \xrightarrow{p} \sigma_\theta^2$, then $T \xrightarrow{D} \text{Normal}(0, 1)$

(Theorem 6.21, Bierens 2004). Hence, the confidence interval of θ is given by:

$$\hat{\theta} - z_\delta SE\left(\hat{\theta}\right) \leq \theta \leq \hat{\theta} + z_\delta SE\left(\hat{\theta}\right),$$

$$\Rightarrow \quad \hat{\theta}\left(1 - z_\delta \hat{\theta}^{-\alpha}\sqrt{\frac{\sum_{n=1}^{N}\left(\eta_i - \bar{\eta}\right)^2}{\alpha^2 N\left(N-1\right)}}\right) \leq \theta \leq \hat{\theta}\left(1 + z_\delta \hat{\theta}^{-\alpha}\sqrt{\frac{\sum_{n=1}^{N}\left(\eta_i - \bar{\eta}\right)^2}{\alpha^2 N\left(N-1\right)}}\right)$$

where z_δ is the critical value with confidence level of $(1-\delta)\%$.

9 Conclusion

In this paper, a two-parameter class of multidimensional welfare indices is developed that is sensitive to inter-personal inequality. The concern for inter-personal inequality in the multidimensional framework can take two forms. The first one pertains to the spread of each dimensional achievement across the population, as would be reflected in the multidimensional version of the usual Lorenz criterion (Kolm 1977). The second one regards correlation across dimensions, reflecting the key observation that inter-dimensional association may alter evaluation of individual as well as overall inequality (Atkinson and Bourguignon 1982). The class of indices developed in this paper is shown to be sensitive to both these forms of multidimensional inequality under proper restrictions on the parameters.

The functional form of the class is characterized by certain reasonable axioms and it is shown that the class is based on generalized means, making it friendly to empirical application for the simple functional structure. All indices in the class are subgroup consistent implying that increase in welfare of one group leading to increase in the overall welfare, while that of the other group remained unaltered. The class is broader in the sense that it includes few other existing classes of welfare indices, such as the ones developed by Bourguignon (1999) and Foster, Lopez-Calva, and Szekely (2005). After a welfare evaluation is obtained by summarizing the sample dataset of achievements, an obvious concern is raised about the reliability of the estimate of the evaluation. For this purpose, the paper develops appropriate statistical tests.

Although, the class of indices has several nice features, many problems in multidimensional welfare measurement still remain untouched. First, by construction, all indices in this class still assume the same degree of substitution between each pair of dimensions. Alternatively speaking, it is assumed that all dimensions are either substitutes or complements to each other. It is not possible for the proposed class of indices to treat few dimensions to be substitutes while the rest as complements. It would be interesting to find a way to be able to incorporate different degrees of substitution between dimensions.

Secondly, this paper considers a type of association increasing transfer following Boland and Proschan (1988). This concept is preferred over the principle called *correlation increasing switch* (CIS) defined by Bourguignon and Chakravarty (2003), because the interpretation of the CIS is not reasonably transparent. The CIS argues that multidimensional correlation increases if the correlation between any two marginal distribution increases, but does consider what happens to the other pair-wise correlations. The definition of association increasing transfer, on the other hand, is indeed more transparent. However, the concept of association increasing transfer is the only notion that is used in this paper. Further research is required in this area to explore other possible ways of increasing association and verify whether the class of indices is sensitive to them. Finally, in this paper, we do not propose any idea about how to choose the weight vectors and parameter values, and assume that they are determined normatively by policy makers.

There are various challenges while measuring social welfare. The first is the choice the appropriate dimensions. The second is the collection of reasonable data resembling the proper state of the world. A third challenge is related to the selection of appropriate aggregation method so the data can be summarized properly to evaluate the accurate level of social welfare. The final challenge is the political good will. This paper focuses on the third challenge only and leaves the rest of the challenges to be remedied by other research scholars and policy makers.

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Appendix A: Proof of Theorem 1

Proof. The sufficiency part of the proof is straight forward. It can be easily shown that if the social welfare function is of the form (3), then it satisfies CNT, NM, LH, SP, PRI, MO, and SC. These results follow mostly from the fact that generalized means also satisfy the set of axioms.

Next, we show that the set of axioms enable us to obtain the particular form of the social welfare index in (3), i.e., we prove the necessary part. Consider two distributions of standardize achievements, s and t, with the same population size $N > 2$,. Let us split both distributions into two subgroups of size N_1 and size $N_2 = N - N_1$ such that $s = (s_1, s_2)$ and $t = (t_1, t_2)$. It can be shown that subgroup consistency implies:

$$\Phi_N(s_1, s_2) \geq \Phi_N(t_1, s_2) \Rightarrow \Phi_N(s_1, t_2) \geq \Phi_N(t_1, t_2). \quad \text{(A-1)}$$

Note that axiom MO and axiom CNT ensures $\Phi_N(s)$ to have only one value for any $s \in \mathbb{R}^N_{++}$. If $\Phi_N(s_1, s_2) > \Phi_N(t_1, s_2)$, then by SC, we can never have $\Phi_N(s_1) \leq \Phi_N(t_1)$ and, thus, $\Phi_N(s_1) > \Phi_N(t_1)$. A further application of SC ensures that $\Phi_N(s_1, t_2) > \Phi_N(t_1, t_2)$. Now, we need to show that $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2) \Rightarrow \Phi_N(s_1, t_2) = \Phi_N(t_1, t_2)$. First, let $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2)$ and $\Phi_N(s_1) > \Phi_N(t_1)$. However, SC requires that $\Phi_N(s_1) > \Phi_N(t_1) \Rightarrow \Phi_N(s_1, s_2) > \Phi_N(t_1, s_2)$. This is a contradiction. Similarly, it can be shown that $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2)$ and $\Phi_N(s_1) < \Phi_N(t_1)$ does not hold together. Therefore, $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2) \Rightarrow \Phi_N(s_1) = \Phi_N(t_1)$. With further application of SC, it can be shown that $\Phi_N(s_1) = \Phi_N(t_1) \Rightarrow \Phi_N(s_1, s_2) = \Phi_N(t_1, s_2)$.

The formulation in (A-1) is equivalent to the concept of strong separability in the existing literature (Gorman (1968), Blackorby, Primont, and Russel (1978)). Strong separability leads the form of the social welfare index in (2) to be:

$$W(H) = \phi_N\left(\sum_{n=1}^{N} \Delta_{N,n}(Q(h_{n\cdot}))\right) = \phi_N\left(\sum_{n=1}^{N} \Delta_{N,n}\left[U\left(\sum_{d=1}^{D} V_d(h_{nd})\right)\right]\right); \quad \text{(A-2)}$$

where ϕ_N is continuous and strictly increasing and $\Delta_{N,n} : \mathbb{R}_{++} \to \mathbb{R}$ is continuous.

According to axiom NM, we require that

$$\phi_N\left(\sum_{n=1}^{N} \Delta_{N,n}\left[U\left(\sum_{d=1}^{D} V_d(\theta)\right)\right]\right) = \theta \text{ for any } \theta \in \mathbb{R}_{++}.$$

In other words, we require $U\left(\sum_{d=1}^{D} V_d(\theta)\right) = \theta$ and $\phi_N\left(\sum_{n=1}^{N} \cdot_{N,n}(\theta)\right) = \theta$. As a result, the form that (A-2) takes is:

$$W(H) = \Omega_N^{-1}\left(\sum_{n=1}^{N} b_n \Omega_N V^{-1}\left(\sum_{d=1}^{D} a_d V(h_{nd})\right)\right), \quad \text{(A-3)}$$

where $\phi_N = \Omega_N^{-1}$, $\cdot_{N,n} = b_n \Omega_N V^{-1}$, $V_d = a_d V$, $a_d, b_n \in \mathbb{R}_+ \ \forall n, d$, $\sum_{d=1}^{D} b_n = 1$, and $\sum_{d=1}^{D} a_d = 1$. Both Ω_N and V are strictly increasing and continuous.

Axiom SP requires each person to be anonymous, and, thus, $b_n = 1/N \ \forall n$. The functional form (A-3) becomes:

$$W(H) = \Omega_N^{-1}\left(\frac{1}{N}\sum_{n=1}^{N} \Omega_N V^{-1}\left(\sum_{d=1}^{D} a_d V(h_{nd})\right)\right). \quad \text{(A-4)}$$

Therefore, we express the individual aggregation function as:

$$Q(\cdot) = V^{-1}\left(\sum_{d=1}^{D} a_d V(\cdot)\right)$$

and the standardized achievement aggregation function can be written as:

$$\Phi_N(\cdot) = \Omega_N^{-1}\left(\frac{1}{N}\sum_{n=1}^{N} \Omega_N(\cdot)\right).$$

Following Foster and Székely (2008), axiom LH and axiom PRI result the functional form for the standardized achievement aggregation function to be:

$$\Phi\left(Q\left(h_{1\cdot}\right),\ldots,Q\left(h_{N\cdot}\right)\right)=\begin{cases}\left(\frac{1}{N}\sum_{n=1}^{N}Q\left(h_{n\cdot}\right)^{\beta}\right)^{1/\beta} & \beta\neq 0\\ \left(\prod_{n=1}^{N}Q\left(h_{n\cdot}\right)\right)^{1/N} & \beta=0\end{cases}. \tag{A-5}$$

Finally, we need to derive the functional form for the individual aggregation function. The individual aggregation function $Q(\cdot)$ is a quasi-linear mean (Eichhorn, 1978, p. 32) since V satisfies CNT and MO. As $Q(\cdot)$ satisfies NM, Theorem 2.2.1 of Eichhorn (1978) leads the functional form to be:

$$Q\left(h_{n\cdot}\right)=\begin{cases}\left(\sum_{d=1}^{D}a_d h_{nd}^{\alpha}\right)^{1/\alpha} & \alpha\neq 0\\ \prod_{d=1}^{D}h_{nd}^{a_d} & \alpha=0\end{cases}\quad \forall n=1,\ldots,N. \tag{A-6}$$

Combining (A-5) and (A-6) together, we obtain the class of social welfare index in (3). ■

Appendix B: Proof of Theorem 2

This theorem is proved with the help of Definition B1, Definition B2, Proposition B1, and Corollary B1 as introduced below. For the purpose of the proof, we break down the functional form in (3) into the following three functional forms:

$$\mathcal{W}(\cdot)=\mathcal{F}\left(F\left(G(\cdot)\right)\right). \tag{B-1}$$

Based on different values of α and β, the various functional forms are summarized in Table B1.

Table B1: Different Functional Forms of $\mathcal{W}$ Break Up

	$\mathcal{F}(\cdot)$	$F(\cdot)$	$G(h_{n\cdot})$
Case I: $\alpha\neq 0,\ \beta\neq 0$	$\left(\frac{1}{N}F(\cdot)\right)^{1/\alpha}$	$\sum_{n=1}^{N}G(\cdot)$	$\mu_{\beta}^{\alpha}(h_{n\cdot};a)$
Case II: $\alpha\neq 0,\ \beta=0$	$\left(\frac{1}{N}F(\cdot)\right)^{1/\alpha}$	$\sum_{n=1}^{N}G(\cdot)$	$\mu_{0}^{\alpha}(h_{n\cdot};a)$
Case III: $\alpha=0,\ \beta\neq 0$	$\left(F(\cdot)\right)^{1/N}$	$\prod_{n=1}^{N}G(\cdot)$	$\mu_{\beta}(h_{n\cdot};a)$
Case IV: $\alpha=0,\ \beta=0$	$\left(F(\cdot)\right)^{1/N}$	$\prod_{n=1}^{N}G(\cdot)$	$\mu_{0}(h_{n\cdot};a)$.

Next, based on Table B1, we would obtain the restrictions on parameters that enable $\mathcal{W}(\cdot)$ to be strictly association sensitive. However, the next set of results are founded on the lattice theory. Definition B1 introduces strict L-subadditivity, L-superadditivity, and valuation[12].

Definition B1 *For every $M\in\mathbb{N}$, for every $x,y\in\mathbb{R}_{+}^{M}$, (i) any function $\mathbf{G}$ is strict L-subadditive if $\mathbf{G}(x\vee y)+\mathbf{G}(x\wedge y)<\mathbf{G}(x)+\mathbf{G}(y)$, (ii) any function $\mathbf{G}$ is strict L-superadditive if $\mathbf{G}(x\vee y)+\mathbf{G}(x\wedge y)>\mathbf{G}(x)+\mathbf{G}(y)$, and (iii) any function $\mathbf{G}$ is a valuation if $\mathbf{G}(x\vee y)+\mathbf{G}(x\wedge y)=\mathbf{G}(x)+\mathbf{G}(y)$.*

The impact of association increasing transfers on $F(\cdot)$ depends on whether $G(\cdot)$ is L-subadditive, L-superadditive, or valuation. It is apparent from Table B1 that there are two distinct

[12] L-subadditive and L-superadditive stands for Lattice subadditive and Lattice superadditive, respectively; whereas, valuation implies both Lattice subadditive and Lattice superadditive.

functional forms for $F(\cdot)$. The first is additive, i.e., $F(\cdot) = \sum_{i=1}^{n} G(\cdot)$, corresponding to $\beta \neq 0$ and the second is multiplicative, i.e., $F(\cdot) = \prod_{i=1}^{n} G(\cdot)$, corresponding to $\beta = 0$. The following proposition, due to Boland and Proschan (1988), is helpful in deriving the impact of accociation increasing transfers on the additive form of $F(\cdot)$.

Proposition B1 *For every* $\mathbf{N} \subset \mathbb{N}$ *and* $N > 1$*, for every* $\mathbf{F}(H) = \sum_{n=1}^{N} \mathbf{G}(h_{n\cdot})$*, and for every* $H_N, H'_N \in \mathcal{H}_N$ *such that* H'_N *is obtained from* H_N *by a finite sequence of association increasing transfers, (i)* $\mathbf{F}(H_N) < \mathbf{F}(H'_N)$ *if and only if* $\mathbf{G}(\cdot)$ *is strict L-subadditive on* $\mathbb{R}_+^D$*, (ii)* $\mathbf{F}(H_N) > \mathbf{F}(H'_N)$ *if and only if* $\mathbf{G}(\cdot)$ *is strict L-superadditive on* $\mathbb{R}_+^D$*, and (iii)* $\mathbf{F}(H_N) = \mathbf{F}(H'_N)$ *if and only if* $\mathbf{G}(\cdot)$ *is a valuation on* $\mathbb{R}_+^D$*.*

Proof. See Proposition 2.5 (b) in Boland and Proschan (1988). ■

The following corollary derives the impact of association increasing transfer on $F(\cdot)$ when it takes the multiplicative form.

Corollary B1 *For every* $\mathbf{N} \subset \mathbb{N}$ *and* $N > 1$*, for every* $\mathbf{F}(H_N) = \prod_{n=1}^{N} \mathbf{G}(h_{n\cdot})$*, and for every* $H_N, H'_N \in \mathcal{H}_N$ *such that* H'_N *is obtained from* H_N *by a finite sequence of association increasing transfers, (i)* $\mathbf{F}(H_N) < \mathbf{F}(H'_N)$ *if and only if* $\log \mathbf{G}(\cdot)$ *is strict L-subadditive on* $\mathbb{R}_+^D$*, (ii)* $\mathbf{F}(H_N) > \mathbf{F}(H'_N)$ *if and only if* $\log \mathbf{G}(\cdot)$ *is strict L-superadditive on* $\mathbb{R}_+^D$*, and (iii)* $\mathbf{F}(H_N) = \mathbf{F}(H'_N)$ *if and only if* $\log \mathbf{G}(\cdot)$ *is a valuation on* $\mathbb{R}_+^D$*.*

Proof. Let $H_N, H'_N \in \mathcal{H}_N$ for any $\mathbf{N} \subset \mathbb{N}$ and $N > 1$. There exist persons $n_1, n_2 \in \mathbf{N}$ and $n_1 \neq n_2$, such that $h_{n_1\cdot} \ngtrless h_{n_2\cdot}$. H'_N is obtained from H_N by a sequence of association increasing transfers. From Proposition 1, we know that $\log \mathbf{F}(H'_N) < \log \mathbf{F}(H_N)$ if and only if $\log \mathbf{G}(\cdot)$ is strict L-subadditive. As $\log \mathbf{F}(\cdot)$ is a monotonic transformation of $\mathbf{F}(\cdot)$, it follows that $\mathbf{F}(H'_N) < \mathbf{F}(H_N)$. The other two parts can be proved in identical fashion. ■

From Proposition B1 and Corollary B1, it is apparent that $F(\cdot)$ reacts differently to association increasing transfers under different circumstances. The remaining task is to implement the proposition and the corollary on the proposed class of indices to obtain the proper restrictions on α and β. However, we need to figure out the restrictions on α and β beforehand that enables $G(\cdot)$ to be strictly L-subadditive, strictly L-superadditive, and valuation. The following definition is helpful for this purpose.

Definition B2 *For any twice differentiable function* $\mathbf{G} : R_{++}^D \to R_+$*, (i) strict L-subadditivity requires all cross partial derivatives to be negative, i.e.* $\partial^2 \mathbf{G}(h_{n\cdot}) / \partial h_{nd_1} \partial h_{nd_2} < 0\ \forall d_1 \neq d_2$*; (ii) strict L-superadditivity requires* $\partial^2 \mathbf{G}(h_{n\cdot}) / \partial h_{nd_1} \partial h_{nd_2} > 0\ \forall d_1 \neq d_2$*; (iii) valuation requires* $\partial^2 \mathbf{G}(h_{n\cdot}) / \partial h_{nd_1} \partial h_{nd_2} = 0\ \forall d_1 \neq d_2$*. (Topkis* 1998)

All functional forms of $G(h_{n\cdot})$ in Table B1 are various forms of generalized means and, thus, are twice differentiable for $h_{nd} \in \mathbb{R}_{++}\ \forall n, d$. Table B2 summarizes the restrictions on α and β that allow $G(\cdot)$ and $\log G(\cdot)$ to satisfy strict L-subadditivity, strict L-superadditivity, and valuation. Using the restrictions on the parameters in Table B2 and based on Proposition B1 and Corollary B1, we prove Theorem 2.

Proof of Theorem 2. At first, we prove the sufficient conditions. Let $H_N, H'_N \in \mathcal{H}_N$ for an arbitrary $N \in \mathbb{N}$ and H'_N is obtained from H_N by an association increasing transfer. Let us first consider the situation when $\beta \in \mathbb{R}$ and $\alpha \neq 0$. We summarize parameter α,β, and vector a by θ. By (B-1), we have $\mathcal{W}(H_N; \theta) = \mathcal{F}(F(H_N)) = (F(H_N)/N)^{1/\alpha}$ and $F(\cdot) = \sum_{n=1}^{N} G(\cdot)$.

Table B2: Restrictions on α and β

Strict L-subadditive	Strict L-superadditive	Valuation
$\alpha < 0$ & $\alpha > \beta$	$\alpha < 0$ & $\alpha < \beta$	$\alpha = \beta$
$\alpha > 0$ & $\alpha < \beta$	$\alpha > 0$ & $\alpha > \beta$	
$\alpha = 0$ & $\beta > 0$	$\alpha > 0$ & $\beta = 0$	
	$\alpha < 0$ & $\beta = 0$	
	$\alpha = 0$ & $\beta < 0$	

(i) For $\alpha > 0$ & $\alpha < \beta$, $G(\cdot)$ is L-subadditive. Therefore, $F(H'_N) < F(H_N)$ by Proposition B1, which implies $\mathcal{W}(H'_N;\theta) < \mathcal{W}(H_N;\theta)$.

For $\alpha < 0$ & $\alpha < \beta$, and $\beta \neq 0$, $G(\cdot)$ is L-superadditive. Therefore, $F(H'_N) > F(H_N)$ by Proposition B1, which implies $\mathcal{W}(H'_N;\theta) < \mathcal{W}(H_N;\theta)$.

For $\alpha < 0$ & $\beta = 0$, $G(\cdot)$ is L-superadditive. Therefore, $F(H'_N) > F(H_N)$ by Proposition B1, which implies $\mathcal{W}(H'_N;\theta) < \mathcal{W}(H_N;\theta)$.

Let us consider, now, the situation when $\alpha = 0$ & $\beta > 0$. We have $\mathcal{W}(H_N;\theta) = \mathcal{F}(F(H_N)) = (F(H_N))^{1/n}$ and $F(\cdot) = \prod_{i=1}^{N} G(\cdot)$. In this situation, $\log G(\cdot)$ is L-subadditive. Therefore, $F(H'_N) < F(H_N)$ by Corollary B2 and, thus, $\mathcal{W}(H'_N;\theta) < \mathcal{W}(H_N;\theta)$.

Hence, if $\alpha < \beta$, $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies SDIA.

(ii) Now, we are going to prove that $\mathcal{W}(\cdot)$ satisfies SIIA if $\alpha > \beta$.

First, consider the situation when $\alpha < 0$ & $\alpha > \beta$. In this situation, $G(\cdot)$ is L-subadditive and $F(H'_N) < F(H_N)$ by Proposition B1. Thus, $\mathcal{W}(H'_N;\theta) > \mathcal{W}(H_N;\theta)$.

Next, consider the situation, when $\alpha > 0$ & $\alpha > \beta$. In this situation, $G(\cdot)$ is L-superadditive and $F(H'_N) > F(H_N)$ by Proposition B1. Thus, $\mathcal{W}(H^{N\prime};\theta) > \mathcal{W}(H_N;\theta)$.

In the third situation, we have $\alpha > 0$ & $\beta = 0$. In this situation, $G(\cdot)$ is L-superadditive and $F(H'_N) > F(H_N)$ by Proposition B1. Thus, $\mathcal{W}(H'_N;\theta) > \mathcal{W}(H_N;\theta)$.

Finally, we have the situation when $\alpha = 0$ & $\beta < 0$. In this situation, $\log G(\cdot)$ is L-superadditive and $F(H') > F(H)$ by Corollary B1. Thus, $\mathcal{W}(H'_N;\theta) > \mathcal{W}(H_N;\theta)$.

Hence, for $\alpha > \beta$, $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies SIIA.

The next part is important for the proof of the necessary condition - it is proved that $\mathcal{W}(H_N;\theta)$ satisfies both *WDIA* or *WIIA* when $\beta = \alpha$. First, consider the situation when $\beta = \alpha \neq 0$. We have $\mathcal{W}(H_N;\theta) = \left(\frac{1}{n}F(\cdot)\right)^{1/\alpha}$, but $G(\cdot)$ is a valuation. Therefore, $F(H'_N) = F(H_N)$ by Proposition B1 and, thus, $\mathcal{W}(H'_N;\theta) = \mathcal{W}(H_N;\theta)$.

Second, consider the situation when $\beta = \alpha = 0$. We have $\mathcal{W}(H_N;\theta) = (F(\cdot))^{1/n}$ and, this time, $\log G(\cdot)$ is a valuation. Therefore, $F(H'_N;\theta) = F(H_N;\theta)$ by Corollary B1 and, thus, $\mathcal{W}(H'_N;\theta) = \mathcal{W}(H_N;\theta)$.

Hence, for $\alpha = \beta$, $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies both *WDIA* or *WIIA*.

Now, we are ready to prove the necessary conditions. First of all, $\alpha \not< \beta \Rightarrow \alpha > \beta$ or $\alpha = \beta$, which, in turn, implies that $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies *SIIA* or both *WDIA* or *WIIA*, but does not satisfy *SDIA*.

Secondly, $\alpha \not> \beta \Rightarrow \alpha < \beta$ or $\alpha = \beta$, which in turn implies that $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies *SDIA* or both *WDIA* or *WIIA*, but does not satisfy *SIIA*.

Finally, $\alpha \neq \beta \Rightarrow \alpha > \beta$ or $\alpha < \beta$, which in turn implies that $\mathcal{W}(H_N; a, \alpha, \beta)$ satisfies *SDIA* or *SIIA* but does not satisfy both *WDIA* or *WIIA*. ■

Appendix C: Proof of Theorem 3

The proof of Theorem 3 is partly based on Proposition C1.

Proposition C1 *For every* $\mathbf{N} \subset \mathbb{N}$, *for every* $W(H) = \Phi(Q(h_{1\cdot}), \ldots, Q(h_{N\cdot}))$, *for every* $H'_N, H_N \in \mathcal{H}_N$, *and for every bistochastic matrix* B *such that* $H'_N = BH_N$, *if* $\Phi(\cdot)$ *is non-decreasing and quasi-concave and* $Q(\cdot)$ *is concave then* $W(H'_N) \geq W(H_N)$.

Proof. See Theorem 4 and Theorem 5 in Kolm (1977). ■

Next, we provide the proof of Theorem 3.

Proof of Theorem 3. Let H'_N,$H_N \in \mathcal{H}_N$ and H'_N is obtained from H_N by common smoothing such that $H'_N = BH_N$. According to Proposition C2, if $\Phi(\cdot)$ is non-decreasing and quasi-concave and $Q(\cdot)$ is concave, then $W(H'_N) \geq W(H_N)$. In the formulation of $\mathcal{W}(\cdot)$, $\Phi(\cdot) = \mu_\alpha(\cdot)$ and $Q(\cdot) = \mu_\beta(\cdot)$. From the properties of generalized means, $Q(\cdot)$ is concave if $\beta \leq 1$ and $\Phi(\cdot)$ is quasi-concave if $\alpha \leq 1$. However, for $\alpha = \beta = 1$, $\mathcal{W}(\cdot)$ does not satisfy SICS since $\mathcal{W}(H'_N) = \mathcal{W}(H_N)$. Thus, $\mathcal{W}(\cdot)$ satisfies SICS if $\alpha, \beta \leq 1$ and $\alpha = \beta \neq 1$ and $\mathcal{W}(\cdot)$ satisfies *WICS* if $\alpha, \beta \leq 1$.

Next, we prove the necessary conditions. First, suppose, $\alpha > 1$. For any $N \in \mathbb{N}$, consider $H_N \in \mathcal{H}_N$ such that $h_{\cdot d} = \mathbf{h} \in \mathbb{R}^N_{++}$ $\forall d$. For every weight vector $a \in \mathbb{R}^D_+$ and for every $\beta \in \mathbb{R}$, the individual aggregation function $\mu_\beta(\cdot; a)$ yields the standardized achievement vector $\mathbf{h}$. Finally, for every $\bar{a} = \mathbf{1}_N/N$, we obtain $\mathcal{W}(H_N; a, \alpha, \beta) = \mu_\alpha(\mathbf{h}; \bar{a})$. Construct an achievement matrix $H'_N = BH_N$, where B is any bistochastic matrix. By construction, $h'_{\cdot d} = \mathbf{h}' \in \mathbb{R}^N_{++}$ $\forall d$. Again, the individual aggregation function yields $\mathbf{h}'$ as the vector of standardized achievements such that $\mathbf{h}' = B\mathbf{h}$. Therefore, $\mathcal{W}(H_N; \alpha, \beta, a, \bar{a}) > \mathcal{W}(H'_N; \alpha, \beta, a, \bar{a})$ since $\alpha > 1$ and axiom SICS is violated.

Second, suppose $\beta > 1$, $n = d = 2$ and $a = \bar{a} = (0.5, 0.5)$. Let the achievement vectors of the first and the second persons be (h_{11}, h_{12}) and (h_{21}, h_{22}), respectively, such that $h_{11} = h_{22}$ and $h_{12} = h_{21}$. We denote the achievement matrix by H_0. Thus, for every $\alpha \in \mathbb{R}$, $\mathcal{W}(H_0; \alpha, \beta, a, \bar{a}) = (0.5h_{11}^\beta + 0.5h_{12}^\beta)^{1/\beta}$. Construct $\bar{H}_0 = \bar{B}H_0$, where $\bar{B} = \mathbf{1}_{22}/2$. In this situation, for every $\alpha \in \mathbb{R}$, $\mathcal{W}\left(\bar{H}_0; \alpha, \beta, a, \bar{a}\right) = 0.5(h_{11} + h_{12})$. If $\beta > 1$, then $\mathcal{W}(H_0; \alpha, \beta, a, \bar{a}) > \mathcal{W}\left(\bar{H}_0; \alpha, \beta, a, \bar{a}\right)$. Therefore, axiom *SICS* is violated.

Finally, suppose $\alpha = \beta = 1$. Then $\mathcal{W}(H_N; 1, 1, a, \bar{a}) = \mu(\mu(h_{1\cdot}; a), \ldots, \mu(h_{N\cdot}; a); \bar{a})$ for every $N \in \mathbb{N}$ and for every $H_N \in \mathcal{H}_N$. Construct an achievement matrix $H'_N = BH_N$, where B is any bistochastic matrix. By construction, $\mu(h_{\cdot d}) = \mu(h'_{\cdot d})$ $\forall d$. A little manipulation can show that

$$\mu(\mu(h_{1\cdot}; a), \ldots, \mu(h_{N\cdot}; a); \bar{a}) = \mu(\mu(h_{\cdot 1}; \bar{a}), \ldots, \mu(h_{\cdot D}; \bar{a}); a).$$

Therefore, for every $a \in \mathbb{R}^D_+$

$$\begin{aligned} & \mu(\mu(h_{\cdot 1}), \ldots, \mu(h_{\cdot D}); a) = \mu\left(\mu\left(h'_{\cdot 1}\right), \ldots, \mu\left(h'_{\cdot D}\right); a\right) \\ \Rightarrow \quad & \mathcal{W}(H; 1, 1, a, \bar{a}) \not< \mathcal{W}\left(H'; 1, 1, a, \bar{a}\right). \end{aligned}$$

Hence, axiom *SICS* is violated. This completes the proof. ■