Multidimensional Inequality

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Measurement of Multidimensional Inequality

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    - Pros and cons of these indices

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- N persons and D dimensions

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  - For two groups of size N_1 and N_2 such that $N_1 + N_2 = N$, $\mathrm{I}(\mathrm{X}) = f\left(\mathrm{I}(\mathrm{X}_{N_1}), \mathrm{I}(\mathrm{X}_{N_2}), \bar{\mathrm{X}}_{N_1}, \bar{\mathrm{X}}_{N_2}, N_1, N_2\right)$, where $\bar{\mathrm{X}}_{N_1}$ and $\bar{\mathrm{X}}_{N_2}$ are the mean vectors of X_{N_1} and X_{N_2}

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- Continuity (CONTN). I(H) does not change abruptly due to a change in any of the elements in H

Axioms Sensitive to Inequality Across Persons

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- Example: $X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix}$, $\bar{X} = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ \mathbf{0.35} & \mathbf{0.35} & \mathbf{0.6} \\ \mathbf{0.35} & \mathbf{0.35} & \mathbf{0.6} \end{bmatrix}$

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- Correlation Increasing Transfer (CIT). Y is derived from X by a correlation increasing transfer if, for some rows n' and n'', $n' < n''$, $y_{n'} = x_{n'} \wedge x_{n''}$, $y_{n''} = x_{n'} \vee x_{n''}$, and $y_n = x_n$ for all $n \notin \{n', n''\}$, where $x \wedge y = (\min\{x_{n'1}, y_{n'1}\}, ..., \min\{x_{n''1}, y_{n''1}\})$ and $x \vee y = (\max\{x_{n'1}, y_{n'1}\}, ..., \max\{x_{n''1}, y_{n''1}\})$ (Boland and Proschan 1988)

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- Note - For indices with two stage aggregation approach, if the aggregation takes place first across persons and then across dimensions, indices do not satisfy CIT

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- Inequality increases with correlation when $\alpha < \beta$

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- Create $\mathbf{h} = (0.5, 0.5, 0.5)$
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- Problems: role of inequality aversion parameter is not clear

Maasoumi Index (1986, 1999)

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- The second stage is a generalized entropy

$$\mathrm{I}_M = \begin{cases} \frac{1}{\alpha(1-\alpha)} \frac{1}{N} \sum_{i=1}^{n} \left(1 - \left(\frac{U_n}{\bar{S}} \right)^{\alpha} \right) & \text{for } \alpha \neq 0, 1. \\ \frac{1}{N} \sum_{i=1}^{n} \log \left(\frac{\bar{S}}{U_n} \right) & \text{for } \alpha = 0 \\ \frac{1}{N} \sum_{i=1}^{n} \frac{U_n}{\bar{S}} \log \left(\frac{U_n}{\bar{S}} \right) & \text{for } \alpha = 1 \end{cases}$$

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$$\mathrm{I}_M = \begin{cases} \frac{1}{\alpha(1-\alpha)}\frac{1}{N}\sum_{i=1}^{n}\left(1-\left(\frac{U_n}{\bar{S}}\right)^{\alpha}\right) & \text{for } \alpha \neq 0, 1. \\ \frac{1}{N}\sum_{i=1}^{n}\log\left(\frac{\bar{S}}{U_n}\right) & \text{for } \alpha = 0 \\ \frac{1}{N}\sum_{i=1}^{n}\frac{U_n}{\bar{S}}\log\left(\frac{U_n}{\bar{S}}\right) & \text{for } \alpha = 1 \end{cases}$$

- $\bar{S} = \frac{1}{N}\sum_{i=1}^{N} U_n$

Maasoumi Index (1986, 1999)

- A two stage procedure
- The first stage is a generalized mean

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- The second stage is a generalized entropy

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  - $\bar{S} = \frac{1}{N} \sum_{i=1}^{N} U_n$
- Problems: Not sure what restrictions on parameter satisfies different transfer properties

Tsui Index (1995, 1999)

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$$I_{TRI} = 1 - \left[\frac{1}{N} \sum_{n=1}^{N} \left(\prod_{d=1}^{D} \left(\frac{x_{nd}}{\mu_d} \right)^{a_d} \right) \right]^{1 / \sum_{i=1}^{D} a_d}$$

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- Unlike Maasoumi, these indices had parameter specification to satisfy transfer.
- Problem: Tsui parameters are not interpretable.

Multidimensional Generalized Gini Indices

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- Limitation: the order of aggregation makes I_{GW} to be not strictly sensitive to correlation among dimensions

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  - First stage: generalized mean across dimensions.

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