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UNIVERSITY OF OXFORD

Summer School on Multidimensional Poverty

8 July – 20 July 2013

The George Washington University

Washington DC

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Unidimensional Poverty Measurement

James Foster

GWU and OPHI

Main Sources of this Lecture

- Foster and Sen (1997), Annexe of "On Economic Inequality".
- Foster (2006) "Poverty Indices"
- Foster, Seth, Lokshin, Sajaia (2013)
- There are others: please see the reading list.

Preliminaries

Preliminaries

- Single dimensional achievement
  - Income, Expenditure, Calories
- Achievements of a society or a country can be represented by a vector or a distribution
- Unit of analysis may be individual or household

Preliminaries

Achievement Vector

Suppose there are four persons in a society with incomes \$9, \$4, \$15 and \$8

- Then $x = (9, 4, 15, 8)$ is a vector representing the incomes of the society

Ordered Achievement Vector

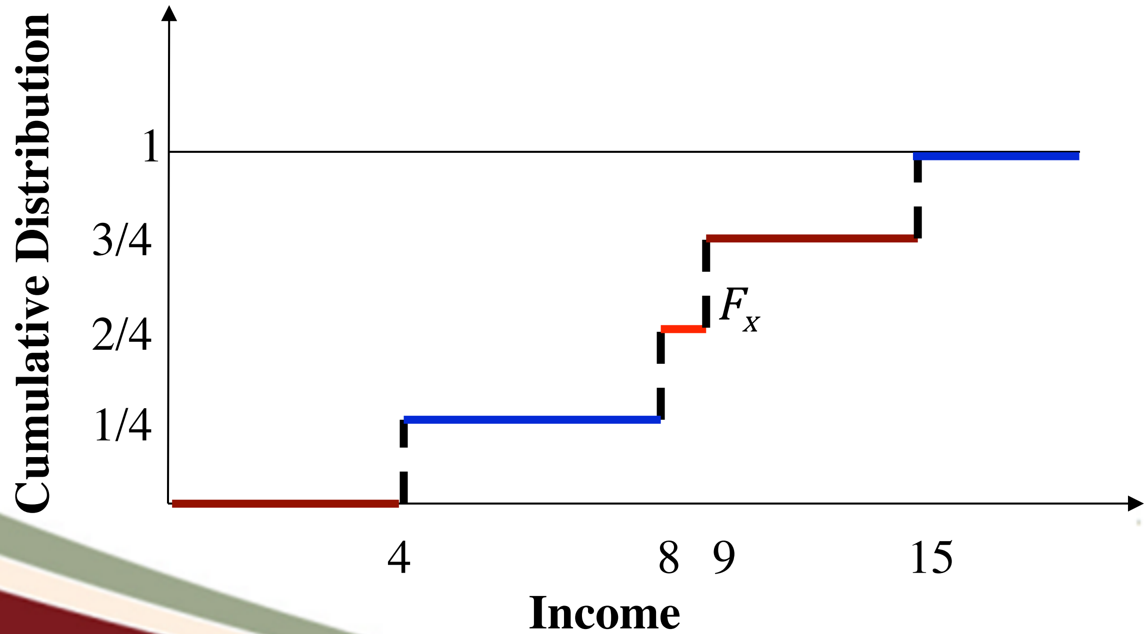
An ordered vector ranks or orders individuals by their achievements

- Ordered vector x^{ord} of x is $x^{ord} = (4, 8, 9, 15)$

Preliminaries

Cumulative Distribution Function (CDF)

The distribution x = (9, 4, 15, 8) can be represented by a cdf. The cdf of distribution x is denoted by F_x

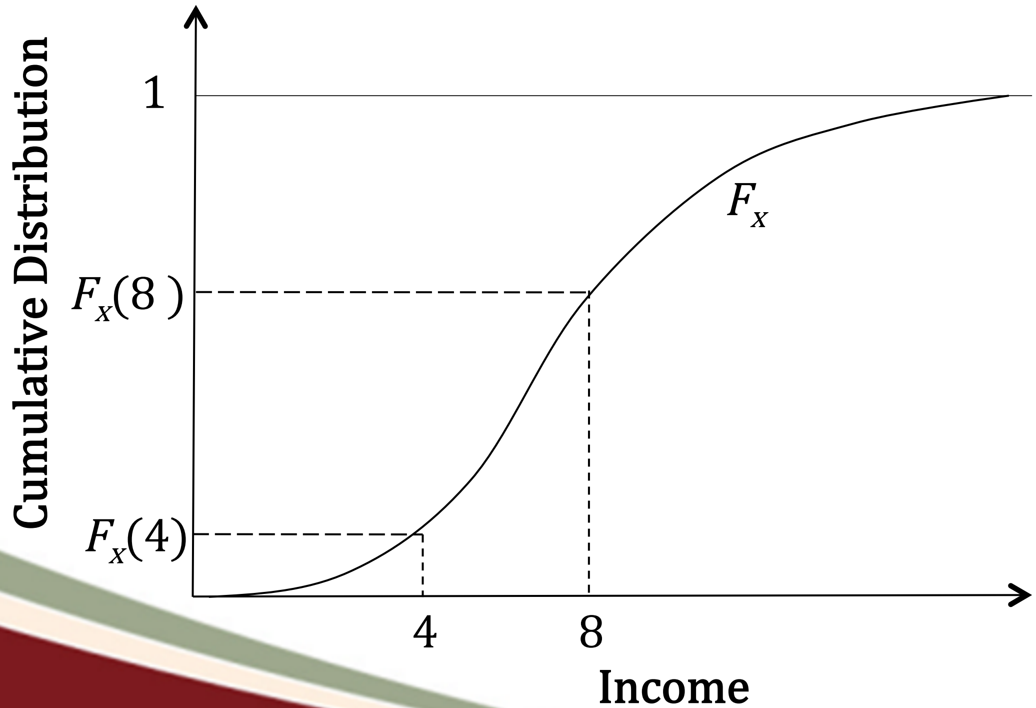


$$F_x = \begin{cases} 0 & \text{for } b<4 \\ 1/4 & \text{for } 4 \leq b < 8 \\ 2/4 & \text{for } 8 \leq b < 9 \\ 3/4 & \text{for } 9 \leq b < 15 \\ 1 & \text{For } b \geq 15 \end{cases}$$

Preliminaries

Cumulative Distribution Function (CDF)

For a society with large population size, a typical cdf looks like



What does a CDF tell us?

It tells us the share of the population having income less than a particular income level

e.g., $F_x(4)$ is the share of the population having income less than $4

Preliminaries

A policy maker is generally interested in the following three aspects of a distribution or a vector

- Size (Welfare), e.g. per-capita income
- Spread (Inequality), e.g. Gini coefficient
- Base (Poverty)
  - Welfare of the population below a certain level of income

In this summer school, we focus on the third aspect

Unidimensional Poverty Measurement

Poverty Measurement

Unidimensional poverty measurement involves two steps (Sen 1976): Identification and Aggregation

Identification: Who is poor?

This step dichotomises the population into *poor* and *non-poor*.

The main tool: poverty line, denoted by z

Person i is poor if $x_i < z$ and is non-poor if $x_i \geq z$

x_i is the i^{th} element of vector x

Types of Poverty Lines

***Absolute Poverty Line* (z_a):** Does **not** depend on the **size** of the entire distribution. Rather usually based on the cost of a set of goods and services considered necessary for having a satisfactory life.
Example: a food poverty line: 2100 calories a day equivalent of consumption expenditure.

***Relative Poverty Line* (z_r):** Depends on the **size** of the entire distribution.
Example: half of the median income.

***Hybrid Poverty Line* (z_h):** Combinations of absolute and relative poverty lines.
Examples: $z = (z_r)^{\rho}(z_a)^{(1-\rho)}$ for $0 < r < 1$ (Foster, 1998);
$z = \max(z_a, \alpha + kM)$, with M being the median income, (Ravallion and Chen, 2009).

Significance of Poverty Line

Enables policy makers to identify a group of people who are subject to different social assistance or 'targeted'

It is 'a benchmark': objective of policy maker is to raise achievements to at least z

- For poverty analysis, achievements of non-poor (above the poverty line) are ignored

Significance of Poverty Line

Censored Distribution of Achievements

Having z as benchmark, allows us to create a censored distribution of x, denoted by x^*, where

$$x_i^* = x_i \text{ if } x_i < z$$

and

$$x_i^* = z \text{ if } x_i \geq z$$

Example: If $z = 10$ and $x = (9, 4, 15, 8)$, then

$$x^* = (9, 4, 10, 8)$$

Second Step: Aggregation

Aggregation: *How poor is the society?*

This step construct an index of poverty summarizing the information in the censored achievement vector x^*.

For each distribution *x* and poverty line z, P(x;z) or $P(x^*)$ indicates the level of poverty in the distribution.

We will adopt an absolute z approach and focus the discussion in terms of the indices

Axioms

Axioms
(Classification of Foster, 2006)

Axiom embody policy: what you do *and do not* want to measure and act upon

- Invariance Axioms
- Dominance Axioms
- Technical Axioms
- Subgroup Axioms

Invariance Axioms

***Symmetry** (Anonymity):* If vector y is obtained from vector x by a *permutation* of incomes and the poverty line remains unchanged, then poverty is unchanged: P(y;z) = P(x;z)

y is obtained from x by a ***permutation*** of incomes if y = Px, where P is a permutation matrix.

Example: z =10, x = (9,4,15,8) and y = (9,15,4,8)

Why is this axiom important?

Invariance Axioms

Permutation Matrix

A square matrix with entries 0 or 1, with rows and columns summing up to one

Example:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Invariance Axioms

***Replication Invariance** (Population Principle)*: If vector y is obtained from vector x by a *replication* and the poverty line remains unchanged, then poverty is unchanged:

$P(y;z) = P(x;z)$

y is obtained from x by a ***replication*** if the incomes in y are simply the incomes in x repeated a finite number of times.

Example: z = 10, x = (9,4,15,8), y = (9,9,4,4,15,15,8,8)

Why is this axiom important?

Invariance Axioms

Focus: If y is obtained from x by *an increment to a non-poor person's income* and the poverty line remains unchanged, then poverty is unchanged: $P(y;z) = P(x;z)$

Example: z = 10, x = (9,4,15,8), y = (9,4,16,8)

Income of the non-poor person increases from $15 to $16, which should not alter a poverty index

Why is this axiom important?

Invariance Axioms

Scale Invariance *(Homogeneity of Zero Degree)*: If all incomes in vector x and the poverty line z are changed by the same *proportion* a>0, then poverty is unchanged: P(ax;az)=P(x;z).

Example: z = 10 and x = (9,4,15,8); if a = 2, then az = 20 and ax = (18,8,30,16)

Why is this axiom important?

Dominance Axioms

Monotonicity: If y is obtained from x by a *decrement* of incomes among the poor and the poverty line remains unchanged, then poverty rises: P(y,z) > P(x,z)

Example 1: z = 10, x = (9,4,15,8); y = (9,4,15,7)

Example 2: z = 10, x = (4,8,9,15); y = (3,8,6,15)

Why is this axiom important?

Dominance Axioms

Transfer: If y is obtained from x by *a progressive transfer* among the poor, then poverty falls:

$P(y;z) < P(x;z)$.

If income is transferred from a person with a higher income to another who has lower income, keeping mean income same, the transfer is called a *progressive transfer*

Example: $z = 10$, $x = (9,4,15,8)$; $y = (9,5,15,7)$

Dominance Axioms

Transfer: *Is there a limit on the amount of transfer in this axiom?*

There is a limit on the amount of transfer. Both post-transfer incomes must be above the lower pre-transfer income

What is the implication of this axiom for non-transferable dimensions?

Technical Axioms

Normalization: As long as everybody is non-poor in vector x for any poverty line z, then $P(x;z) = 0$. In other words, if $\min\{x\} \geq z$, then $P(x;z) = 0$

'Starts' the measure at 0

Example 1: $z = 4$ and $x = (9,4,15,8)$

Example 2: $z = 2$ and $x = (9,4,15,8)$

Technical Axioms

Continuity: For any sequence x^k of distributions converging to x, the respective poverty values $P(x^k;z)$ converge to $P(x;z)$

A technical assumption. It prevents poverty measures from changing abruptly for changes in distribution of achievements

Technical Axioms

Example: x = (9,10,15,8) and z = 10

Suppose the income of the second person begins with $4 and rises to $10

H = 3/4 for all values below $10.

But at x = $10 we have H = 2/4.

H violates continuity.

How about the poverty gap?

Subgroup Axioms

Subgroups

Suppose the population size of vector x is denoted by n(x). Vector x is divided into two population subgroups: x' with population size n(x') and x'' with population size n(x'') such that n(x) = n(x') + n(x'')

Example: Let x = (9,4,15,8), x' = (9,4), x'' = (15,8)

Then, n(x) = 4, n(x') = n(x'') = 2.

Subgroup Axioms

Subgroup Consistency:

If $P(y';z) > P(x';z)$ and $P(y'';z) = P(x'';z)$, where $n(x') = n(y')$ and $n(y'') = n(x'')$, then $P(y;z) > P(x;z)$

Example: Let $z=10$, $x=(9,4,15,8)$, $x'=(9,4)$, $x''=(15,8)$

and $y'=(6,4)$ and $y'' = x''$

Then $P(y';z) > P(x';z)$ for any *monotonicity* poverty measure, and $P(y'';z) = P(x'';z)$, and so one would expect $P(y;z) > P(x;z)$

Subgroup Consistency

Why important?

- Consistent evaluation of poverty reduction programs
- Extension of monotonicity

Monotonicity requires poverty to fall when one person's poverty level is reduced. SC requires aggregate poverty to fall when one group's poverty level is reduced

However, the reduction in group poverty may be accompanied by both increase and fall in individual incomes

Subgroup Axioms

Additive Decomposability: A poverty measure is additive decomposable if:

$$P(x) = \frac{n(x')}{n} P(x') + \frac{n(x'')}{n} P(x'')$$

(Extendable to any number of groups)

One can then calculate the contribution of each group to overall poverty:

$$C(x') = \frac{n(x')P(x')}{nP(x)}$$

Additive decomposability implies subgroup consistency, but the converse does not hold

Subgroup Consistency and Additive Decomposability

P is a continuous, subgroup consistent poverty index if and only if P is a continuous, increasing transformation of a continuous, decomposable poverty index.

(Foster and Shorrocks, 1991)

Poverty Measures

Classification of Measures

Basic Measures

Headcount Ratio

Income Gap Ratio

Poverty Gap Ratio

Advanced Measures

Squared Poverty Gap (Foster-Greer-Thorbecke)

Sen-Shorrocks-Thon Measure

Watts Measure

Clark-Hemming-Ulph-Chakravarty Class of Measures

Basic Poverty Measures

The Headcount Ratio (H)

The most common measure of poverty

Proportion of the population that is poor

Thus, *ranges between 0 and 1*

$H = q/n$

where q is the number of poor and n is the population size.

Example: Let z = 10 and x = (9,4,15,8), then H = 3/4

Basic Poverty Measures

The Headcount Ratio H

What axioms does this measure satisfy?

Satisfies - symmetry, replication invariance, scale invariance, focus, normalization, and subgroup consistency and decomposability

Does not satisfy – monotonicity, transfer, continuity

Policy Implication?

Encourages a policy maker, with limited budget, to assist the marginally poor instead of the severely poor

Basic Poverty Measures

Income Gap Ratio I

The average normalized income gap of the poor; *ranges between 0 and 1*

The normalized gap of the i^{th} poor is $g_i = (z - x_i)/z$.

$$I = S_q\, g_i/q$$

$$= (1/q)S_q(z - x_i)/z = (z - m_p)/z$$

Example: $x=(9,4,15,8)$; $z=10$; $\mu_p=(4+8+9)/3=7$;

Thus, $I = (10 - 7)/10 = 0.3$

Basic Poverty Measures

Income Gap Ratio I

What axioms does this measure satisfy?

Satisfies - symmetry, replication invariance, scale invariance, focus, normalization, monotonicity

Does not satisfy – transfer, continuity, subgroup consistency

Policy Implication?

Counterintuitive: if a poor person's income increases and the person becomes non-poor, poverty increases!

Basic Poverty Measures

Poverty Gap PG

This measure repairs some of the problems of the headcount ratio and income gap ratio

It reports the average normalized gap line using the *censored* distribution x^*

The normalized gap of the i^{th} person is $g_i = (z - x_i^*)/z$.

$PG = S_n\, g_i/n$

$= (z - m^*)/z = \mathbf{H \times I}$

Basic Poverty Measures

Poverty Gap PG

Example: $x=(9,4,15,8)$ and $z=10$.

Then $x^*=(9, 4, \mathbf{10}, 8)$ and $g = (0.1, 0.6, 0, 0.2)$.

So, PG = 0.9/4 = 0.225

Alternatively, where m^* = average of elements in x^*, we have $m^* = 7.75$ and so PG = (10 – 7.75)/10 = 0.225

PG *ranges between 0 and 1*.

1: when everybody is poor with *no income* at all
0: when there is no poor

Basic Poverty Measures

Poverty Gap Ratio PG

What axioms does this measure satisfy?

Satisfies - symmetry, replication invariance, scale invariance, focus, normalization, monotonicity, continuity, subgroup consistency, and decomposability

Does not satisfy – transfer

Policy Implication?

Does not encourage a policy maker to distinguish between a marginally poor and severely poor while assisting

Advanced Poverty Measures

Squared Poverty Gap SG

It reports the average of the *squared* gaps using the *censored* distribution x^*. Also, known as *Foster-Greer-Thorbecke* or P_2 measure

$$SG = S_n (g_i)^2/n$$

Emphasizes the poorest of the poor

Advanced Poverty Measures

Squared Poverty Gap SG

Example: $x=(9,4,15,8)$ and $z=10$

Then $x^*=(9,4,\mathbf{10},8)$ and $g = (0.1,0.6,0,0.2)$

Squares of poverty gap are $g^2 = (0.1^2, 0.6^2, 0^2, 0.2^2) = (0.01, 0.36, 0, 0.04)$.

$SG = 0.41/4 = 0.102$

SG *ranges between 0 and 1*.

1: when everybody is poor with *no income* at all

0: when there is no poor

Advanced Poverty Measures

Squared Poverty Gap SG

What axioms does this measure satisfy?

Satisfies - symmetry, replication invariance, focus, scale invariance, normalization, monotonicity, continuity, transfer, subgroup consistency, and decomposability

Does not satisfy – transfer sensitivity (but that's ok)

This measure can be presented as: $SG = H[I^2 + (1 - I)^2 \times C_p^2]$, where C_p is the coefficient of variation of income across the poor. *Policy implication: care for the severe poor first*

Foster-Greer-Thorbecke (FGT) Class

The FGT class of measures is defined as

$$FGT_a = (1/n)S_n \, g_i^a$$

where a is a parameter and g_i is the normalized income gap of the i^{th} person in x^*

For a = 0, FGT is the Headcount Ratio

For a = 1, FGT is the Poverty Gap Ratio

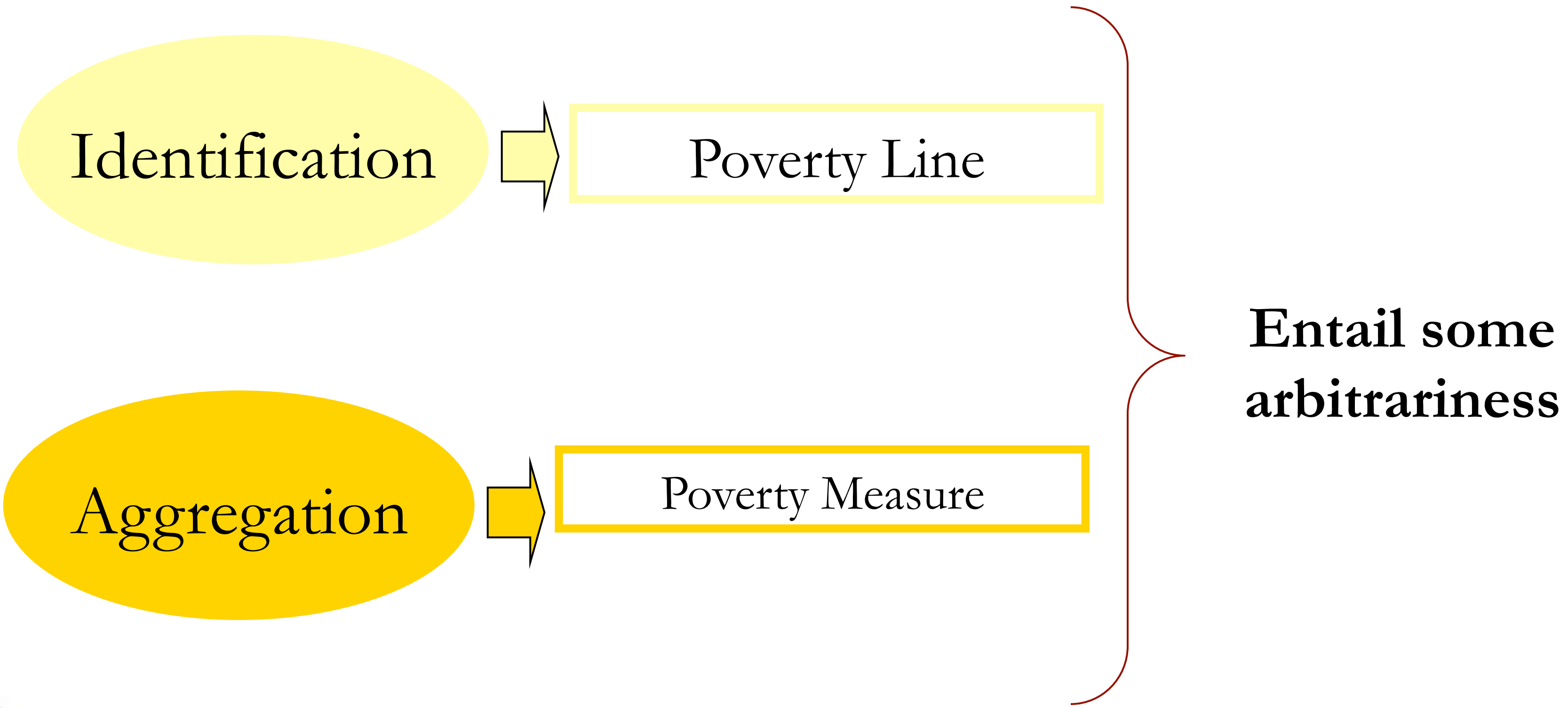
For a = 2, FGT is the Squared Poverty Gap

Unidimensional Dominance

Main Sources of this Lecture

- Foster and Shorrocks (1988)
- Foster and Sen (1997), Annexe of "On Economic Inequality".
- Atkinson (1987)
- There are others: please see the readings list.

When measuring poverty…



Which Poverty Line?

Does the choice of poverty line alter the ranking of distributions? It depends on the distributions

Example: Consider two distributions:

x = (4,8,9,15) and y = (3,6,12,17)

Which distribution has more poverty by H, if z =7?

H(x;z) = 1/4 and H(y;z) = 2/4

Which distribution has more poverty by H, if z =10?

H(x;z) =3/4 and H(y;z) =2/4

Which Measure?

Does the choice of poverty measure alter the ranking of distributions? It depends on the distributions

Example: Consider the same two distributions:

$$x = (4,8,9,15) \text{ and } y = (3,6,12,17)$$

Which distribution has more poverty by H, if z =10?

$$H(x;z) = 3/4 \text{ and } H(y;z) = 2/4$$

Which distribution has more poverty by PG, if z =10?

$$PG(x;z) = 0.225 \text{ and } PG(y;z) = 0.275$$

Dominance Approach

To check robustness directly – it would require checking all poverty lines and measures.

– A tedious, if not impossible task!

Is there any useful tool for these purposes?

– Yes. A tool known as **Stochastic Dominance**

– This is closely linked to poverty ordering, where we rank different distributions

Dominance Approach

Two main types of poverty orderings:

1. Variable-line poverty orderings (focus on the identification step)

2. Variable-measure poverty orderings (address aggregation).

Variable-Line Poverty Orderings (Foster and Shorrocks, 1988)

Main procedure:

1. Choose a measure

2. Identify the condition that two distributions must satisfy so as to be able to say that one has more poverty than the other.

Definition of Poverty Ordering

x**P**y if and only if

$P(x;z) \leq P(y;z)$ for all z and

$P(x;z) < P(y;z)$ for some z

x**P**y means that x has *unambiguously* **less poverty than** y with respect to poverty index P.

FGT Poverty Orderings

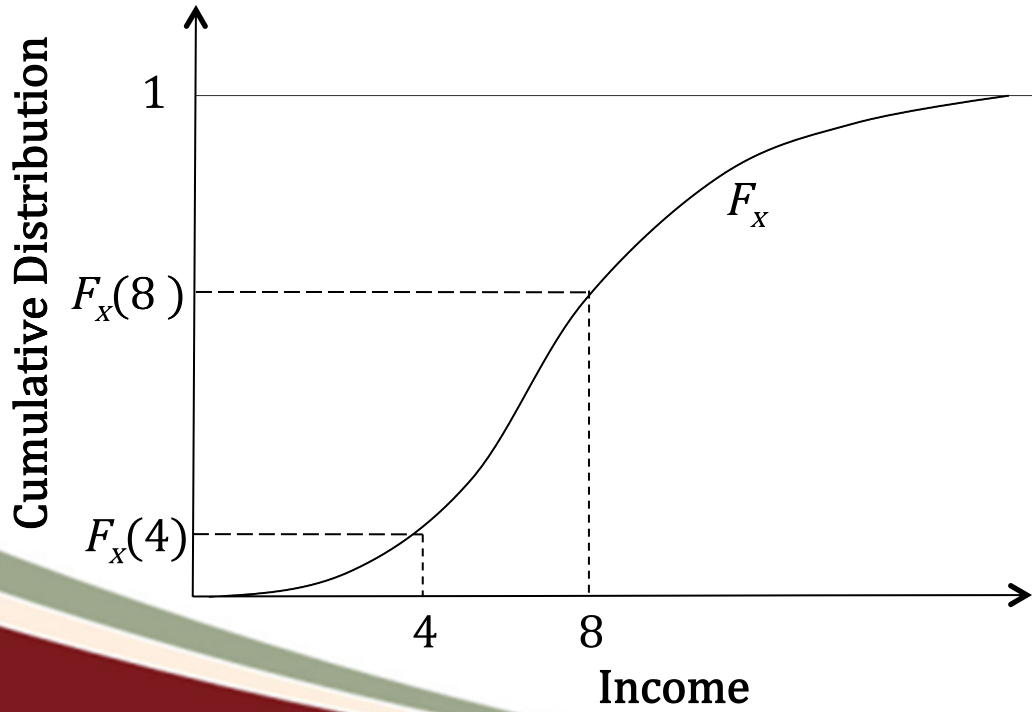
Foster and Shorrocks (1988) developed the conditions of poverty orderings for three members of the FGT family: H, PG and SG

In today's lecture, we only discuss the poverty ordering of the Headcount Ratio H

Poverty Ordering Based on H

Recall the Concept of a CDF

For a society with large population size, a typical cdf looks like



What does a CDF tell us?

It tells us the share of the population having income less than a particular income level

e.g., $F_x(4)$ is the share of the population having income less than $4

Poverty Ordering Based on H

Recall the Concept of a CDF

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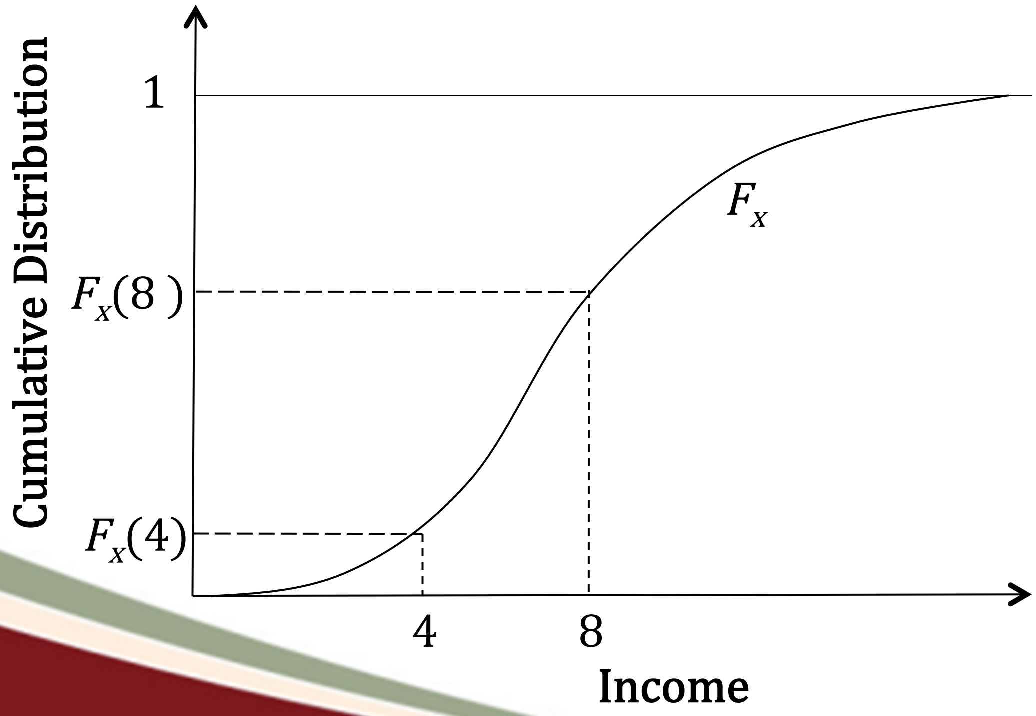


If the poverty line is z=4, then what is H?

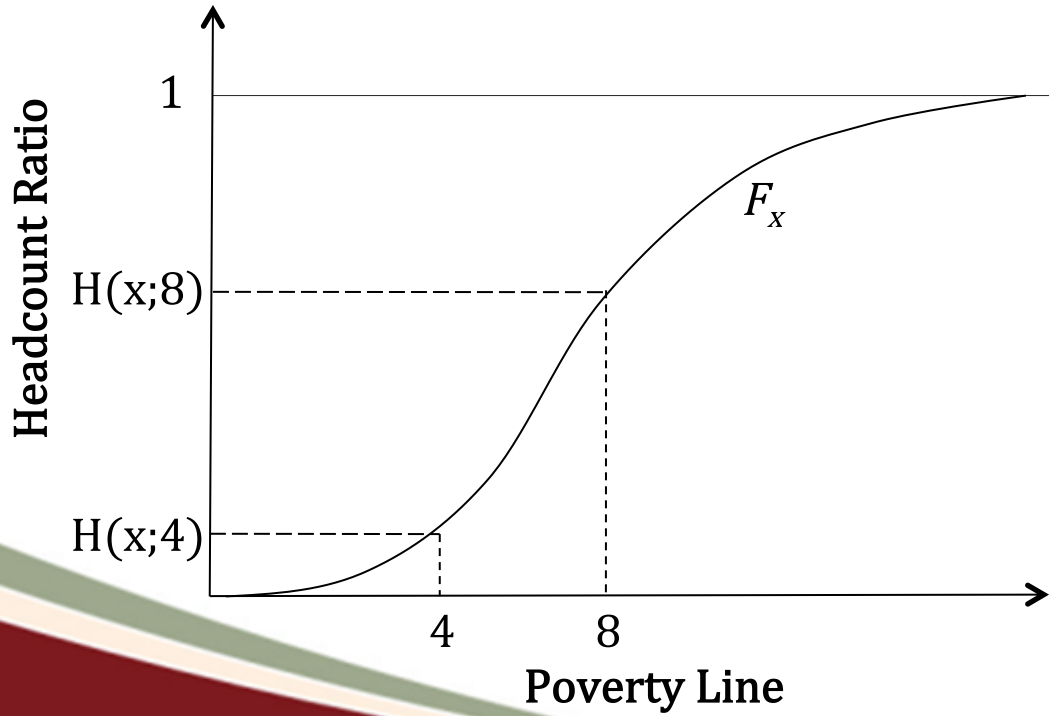
H is equal to $F_x(4)$ in this situation.

The diagram to the left may be presented as:

Poverty Ordering Based on H

Recall the Concept of a CDF

For a society with large population size, a typical cdf looks like



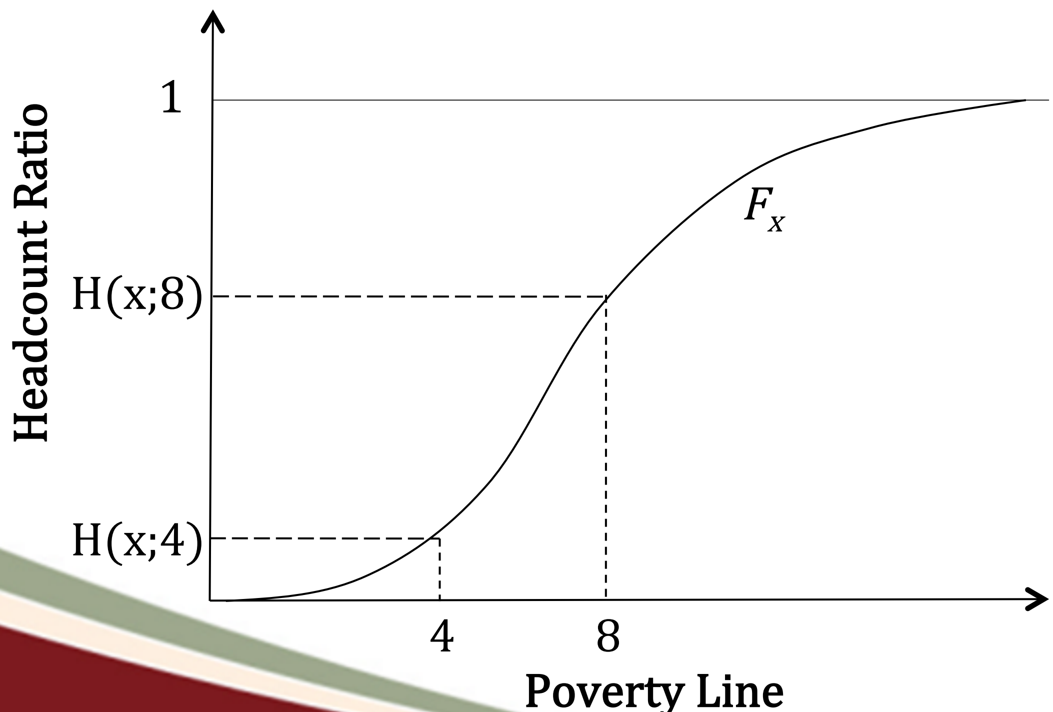
If the poverty line is z=4, then what is H?

H is equal to $F_x(4)$ in this situation.

The diagram to the left may be presented as:

Poverty Ordering Based on H

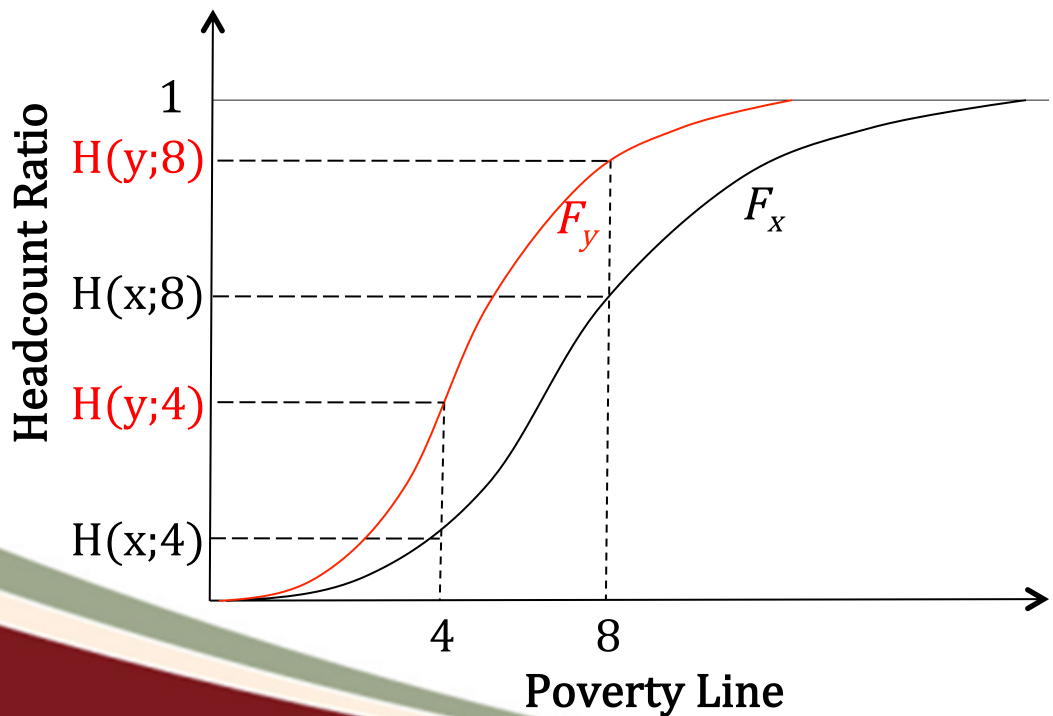
For any poverty line z, the CDF of x gives the headcount ratio



If another cdf F_y lies to the left of F_x then y has a lower headcount ratio than x for every poverty line

Poverty Ordering Based on H

For any poverty line z, the CDF of x gives the headcount ratio



If another cdf F_y lies to the left of F_x then y has a lower headcount ratio than x for every poverty line

First order Stochastic Dominance (FSD)

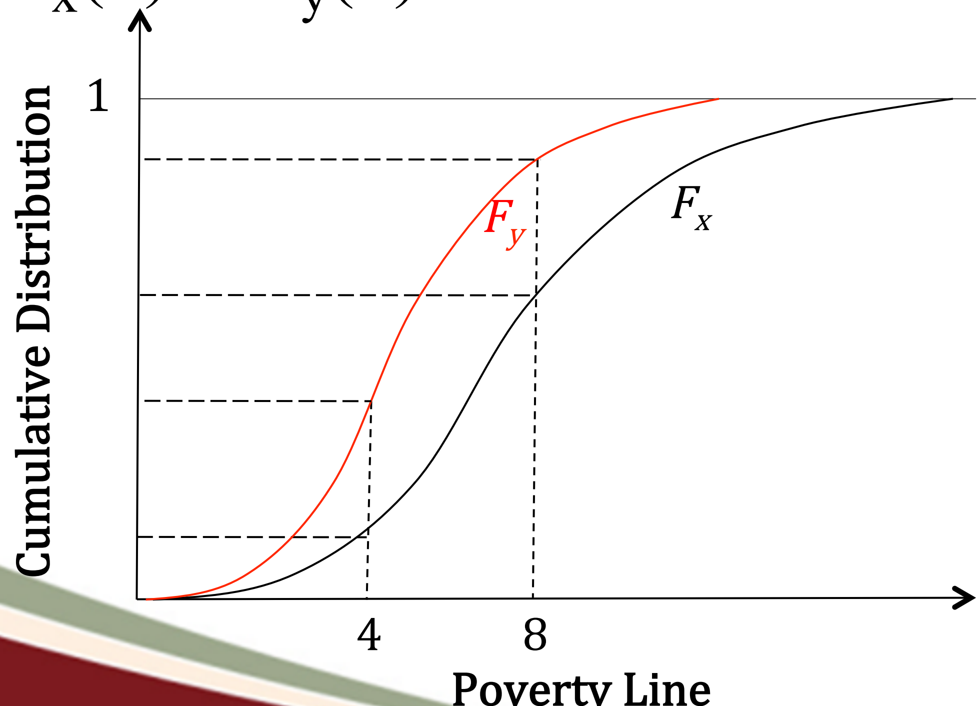
x FSD y

Definition of FSD

For distributions x and y, x FSD y if and only if

$F_x(b) \leq F_y(b)$ for all income levels b and

$F_x(b) < F_y(b)$ for some b



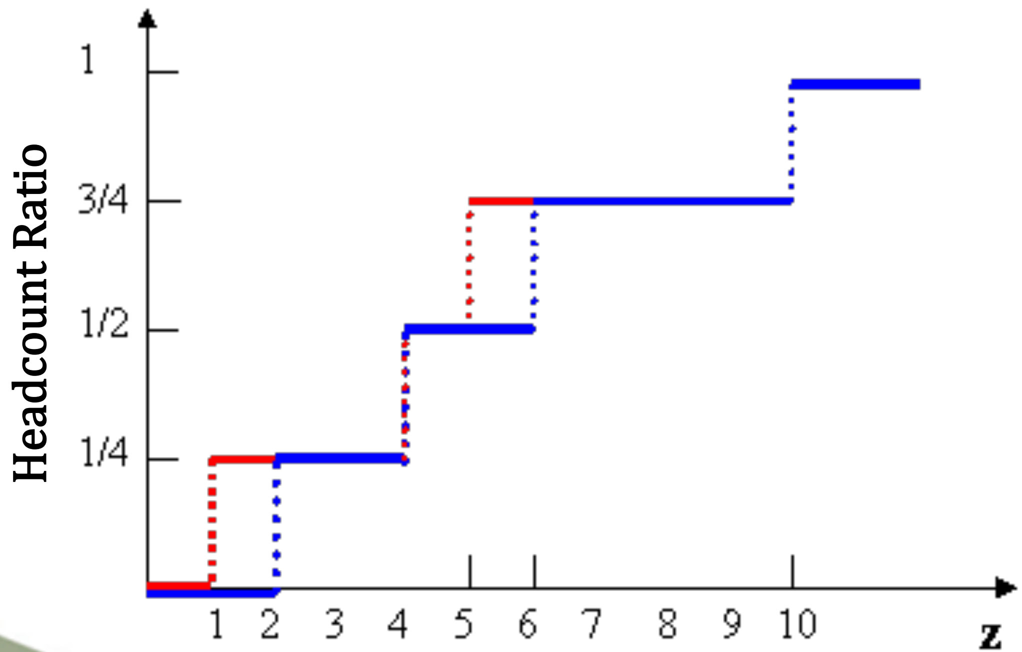
How strong is the FSD result?

If FSD holds, then there is agreement for *all* continuous poverty measures satisfying symmetry, focus, scale and replication invariance and monotonicity for all z.

(Atkinson 1987)

Poverty Ordering Based on H

Example of FSD: Let x=(2,4,6,10) and y=(1,4,5,10)



No part of F_y lies to the right of F_x, and it is strictly to the left for some income

Thus, x FSD y in this case, which means x has unambiguously less poverty than y according to H, (and PG and SG)

Limited Dominance or Robustness Test With Respect to Poverty Line

Limited Range Poverty Orderings

While deciding the precise value of the poverty line may be difficult, agreement is likely to occur on an interval Z.

So now the poverty ordering would be defined as x**P(Z)**y when

$$P(y;z) \geq P(x;z) \text{ for all } z \text{ in } Z$$

$$\text{and} > \text{for some } z \text{ in } Z$$

By restricting the values of z, the obtained ranking P(Z) will be "more complete" than the P ranking (but less general)

Limited Range Poverty Orderings

Indeed, looking at extremely high poverty lines, does not make sense. So now we can set an upper bound z^*, so that the relevant range is $Z^*=(0,z^*)$, and **P*** being the poverty ordering.

One can, then, work with the censored distribution x*, 'ignoring' incomes above z*, i.e., replacing them by z* as before.