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Assessing Deprivation with Ordinal Variables: Depth Sensitivity and Poverty Aversion

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Abstract
The challenges associated with poverty measurement within an axiomatic framework, especially with cardinal variables, have received due attention during the last four decades. However, there is a dearth of literature studying how to meaningfully assess poverty with ordinal variables, capturing the *depth* of deprivations. In this paper, we first propose a class of additively decomposable ordinal poverty measures and provide an axiomatic characterisation using a set of basic foundational properties. Then, in a novel effort, we introduce a set of properties operationalising prioritarianism in the form of different degrees of *poverty aversion* in the ordinal context, and characterise relevant subclasses. We demonstrate the efficacy of our methods using an empirical illustration studying sanitation deprivation in Bangladesh. We further develop related stochastic dominance conditions for all our characterised classes and subclasses of measures. Finally, we elucidate how our ordinal measurement framework is related to the burgeoning literature on multidimensional poverty measurement.

Keywords: Ordinal variables; poverty measurement; precedence to poorer people; Hammond transfer; degree of poverty aversion; stochastic dominance.

JEL Classification: I3, I32, D63, O1

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1 Introduction

Around four decades ago, in an influential article titled 'Poverty: An Ordinal Approach to Measurement', Nobel laureate Amartya Sen proposed an axiomatically derived poverty measure to avoid some shortcomings of the traditionally used headcount ratio (Sen, 1976). Sen's approach was ordinal in the sense that his poverty measures assigned an ordinal-rank weight to each poor person's income, an otherwise cardinal variable. Since then, this seminal article has influenced a well-developed literature on poverty measurement involving cardinal variables within an axiomatic framework (Thon, 1979; Clark et al., 1981; Chakravarty, 1983; Foster et al., 1984; Foster and Shorrocks, 1988a,b; Ravallion, 1994; Shorrocks, 1995).

Distances between the values of cardinally measurable variables are meaningful. By contrast, ordinal variables merely consist of ordered categories and the cardinal distances between these categories are hard to interpret when numerals are assigned to them according to their order or rank.[1] And yet, the practice of using ordinal variables has been on the rise, in both developed and developing countries alike, due to the recent surge of interest in studying deprivation in non-monetary indicators, which are often ordinal in nature (e.g. type of access to basic facilities).[2] Moreover, there may be instances where ordinal categories of an otherwise cardinally measurable variable could have more policy relevance; leading, for example, to focusing on ordered categories of income, nutritional status, or years of education completed, rather than these indicators' cardinal values.

How should poverty be meaningfully assessed with ordinal variables? One straightforward way may be to dichotomise the population into a group of deprived and a group of non-deprived people, and then use the *headcount ratio*. However, this index is widely accused of ignoring the depth of deprivations (Foster and Sen, 1997). For instance, in Sylhet province of Bangladesh between 2007 and 2011, the proportion of population with inadequate sanitation facilities went down from around 70% to nearly 63%; whereas, during the same period, the proportion of people with the worst form of sanitation deprivation ('open defecation') increased significantly, from around 2% to more than 12% (see Table 2 in Section 4).

How can the depth of deprivations be reasonably captured in the case of ordinal variables? The challenges associated with measuring well-being and inequality using an ordinal variable in an axiomatic framework have received due attention during the last few decades (e.g., Mendelson, 1987; Allison and Foster, 2004; Apouey, 2007; Abul Naga and Yalcin, 2008; Zheng, 2011; Kobus and Milos, 2012; Permanyer and D'Ambrosio, 2015; Kobus, 2015; Lazar and Silber, 2013; Yalonetzky, 2013; Gravel et al., 2015). Yet, when assessing poverty, such efforts have not been sufficiently thorough. Bennett and Hatzimasoura (2011), in a rare attempt, showed that indeed we can measure poverty with ordinal variables sensibly, but implicitly ruled out entire classes of well-suited measures (as shown by Yalonetzky, 2012). Moreover, their assessment of depth-sensitivity was restricted to the ordinal version of Pigou-Dalton transfers, thereby missing many other options including the burgeoning use of Hammond transfers (e.g. see Gravel et al., 2015).[3]

[1] Based on the classification of measurement scales by Stevens (1946), whenever numeral scales are assigned to different ordered categories of an ordinal variable according to the orders or ranks of these categories, any 'order-preserving' or monotonic transformation should leave the scale form invariant. See Roberts (1979) for further in-depth discussions. In this paper, by ordinal variables we simply refer to variables with ordered categories, where numeral scales may not have necessarily been assigned to the categories.

[2] For example, as part of the first Sustainable Development Goals, the United Nations has set the target to not only eradicate *extreme monetary poverty*, but also to reduce *poverty in all its dimensions* by 2030. See http://www.un.org/sustainabledevelopment/poverty/ (accessed in April 2017).

[3] We refer to the unidimensional context here. The issue of ordinality has certainly been examined thoroughly in the context of multidimensional poverty measurement (Alkire and Foster, 2011; Bossert et al., 2013; Dhongde et al., 2016; Bosmans et al., 2017). However, even in the multidimensional context, ordinal variables are dichotomised in practice, thereby ignoring the depth of deprivations within

Our paper contributes theoretically to the poverty measurement literature in three ways. First, we axiomatically characterise a class of ordinal poverty measures under a minimal set of desirable properties. Our class consists of measures that are weighted sums of population proportions in deprivation categories, where these weights are referred to as *ordering weights* because their values depend on the order of the categories. Our proposed measures are sensitive to the depth of deprivations (unlike the headcount ratio), are additively decomposable, and are bounded between zero and one.

Second, an adequately designed poverty measure should also ensure that policymakers have proper incentive to prioritise those poorer among the poor in the design of poverty alleviation policies so that *the poorest are not left behind*.[4] In a novel attempt, we operationalise the concept of *precedence to poorer people* by incorporating a *new* form of *degree of poverty aversion* in the ordinal context, reflecting a prioritarian point of view rather than an egalitarian one. Although grounded on prioritarianism, our new form of poverty aversion encompasses, as limiting cases, both previous attempts at sensitising ordinal poverty indices to the depth of deprivations (e.g., Bennett and Hatzimasoura, 2011; Yalonetzky, 2012) as well as current burgeoning approaches to distributional sensitivity in ordinal frameworks based on Hammond transfers (Hammond, 1976; Gravel et al., 2015). We define a range of properties based on this new form of degree of poverty aversion and characterise the corresponding subclasses of ordinal poverty measures. Within our framework, different degrees of poverty aversion merely require setting different restrictions on the ordering weights, preserving the measures' additive decomposability property.

To demonstrate the efficacy of our approach, we present an empirical illustration studying the evolution of sanitation deprivation in Bangladesh. Interestingly, our measures are able to discern the instances where the improvements in overall sanitation deprivation did not necessarily include the poorest.

Since each of our classes and subclasses admits a large number of poverty measures, in our final theoretical contribution we develop related *stochastic dominance conditions* whose fulfilment guarantees the robustness of poverty comparisons to alternative functional forms and measurement parameters. While a few conditions are the ordinal-variable analogue of existing dominance conditions for cardinal variables (Foster and Shorrocks, 1988b), others are themselves a novel methodological contribution to the literature on stochastic dominance with ordinal variables—to the best of our knowledge.

We finally discuss how our proposed class may be applied in the multidimensional context, where multiple variables are used jointly to assess poverty. We show that many well-known additively decomposable multidimensional poverty indices, based on the counting approach (Townsend, 1979; Atkinson, 2003), have the same aggregation expression as our proposed class of ordinal measures.

The rest of the paper proceeds as follows. After providing the notation, we present and axiomatically characterise the class of depth sensitive ordinal poverty measures in Section 2. Section 3 introduces the concept of precedence to poorer people, states the properties related to the degrees of poverty aversion, and characterises the subclass of relevant poverty indices. Section 4 provides an empirical illustration analysing sanitation deprivation in Bangladesh. Then section 5 develops stochastic dominance conditions for the characterised class and subclasses of poverty measures. Section 6 elucidates how our ordinal measurement framework can contribute to the growing literature of multidimensional poverty measurement. Section 7 concludes.

indicators.

[4] Poverty measures may affect the incentives of policymakers during poverty alleviation (Zheng, 1997).

2 A Class of Depth-Sensitive Ordinal Poverty Measures

Suppose there is a *social planner* whose objective is to assess poverty in some well-being dimension in a hypothetical society consisting of $N \in \mathbb{N}$ individuals, where $\mathbb{N}$ is the set of positive integers. Each individual in the society experiences some level of deprivation in the underlying dimension, but the *actual* level of individual deprivation may often be unmeasurable or somehow unobservable to the social planner. Instead, the social planner may merely observe a set of ordered categories. For instance, self-reported health status may only include response categories, such as 'good health', 'fair health', 'poor health', and 'very poor health'. Similarly, there are also instances where the ordinal categories of an otherwise cardinal variable, such as the *Body Mass Index* (BMI) for assessing nutritional status or the *years of schooling completed* for assessing the level of educational attainment, have more policy relevance. Even though the BMI is cardinal, the differences between its cardinal values may not have the same interpretation. According to the World Health Organisation (WHO), BMIs of 15.4 and 15.9 both mean 'severe thinness', but a BMI of 18.4 means 'mild thinness' and a BMI of 18.9 means 'normal weight', despite the same cardinal differences. Notably, a 'severely thin' person is less well nourished than a 'moderately thin' person and both are less well nourished than a 'normal weight' person.

Formally, suppose there is a fixed set of $S \in \mathbb{N}\backslash\{1\}$ ordered categories $c_1, \ldots, c_S$, such that $c_{s-1} \succ_D c_s$ for all $s = 2, \ldots, S$, where $\succ_D$ is a binary, transitive, and reflexive relation whereby category c_{s-1} represents a worse-off situation (or higher deprivation) than category c_s. Thus, c_S is the category reflecting least deprivation and c_1 is the state reflecting highest deprivation. Suppose, for example, a society's well-being is assessed by the education dimension and the observed ordered categories are no education, primary education, secondary education, and higher education, such that no education $\succ_D$ primary education $\succ_D$ secondary education $\succ_D$ higher education. Then, $c_4 =$ higher education and $c_1 =$ no education. We denote the set of all S categories by $\mathbf{C} = \{c_1, c_2, \ldots, c_S\}$ and the set of all categories excluding the category of least deprivation c_S by $\mathbf{C}_{-S} = \mathbf{C} \setminus \{c_S\}$.

Let us denote the actual deprivation level of individual n by $x_n \in \mathbb{R}_+$ for all $n = 1, \ldots, N$, where $\mathbb{R}_+$ is the set of non-negative real numbers. The vector of individual deprivation levels is denoted by $\mathbf{x} = (x_1, \ldots, x_N)$ and the set of all individuals in $\mathbf{x}$ by $\mathbf{N}(\mathbf{x})$. We denote the set of all individual deprivation vectors of population size N by $\mathbf{X}_N$, and the set of all individual deprivation vectors of any population size by $\mathbf{X}$. The social planner does not observe the actual deprivation levels, but observes the deprivation category experienced by each individual. We denote the set of individuals in $\mathbf{x}$ experiencing category c_s by $\Omega_s(\mathbf{x})$, such that $\Omega_s(\mathbf{x}) \cap \Omega_{s'}(\mathbf{x}) = \varnothing$ for all $s \neq s'$ and $\cup_{s=1}^{S} \Omega_s(\mathbf{x}) = \mathbf{N}(\mathbf{x})$. In words, an individual must experience a category and cannot experience more than one category at the same time. Let $N_s(\mathbf{x})$ denote the number of people in $\Omega_s(\mathbf{x})$ and let $p_s(\mathbf{x}) = N_s(\mathbf{x})/N$ denote the proportion of overall population in $\Omega_s(\mathbf{x})$. Then, by definition, $p_s(\mathbf{x}) \geq 0$ for all s and $\sum_{s=1}^{S} p_s(\mathbf{x}) = 1$. We denote the proportions of population in $\mathbf{x}$ across S categories by $\mathbf{p}(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_S(\mathbf{x}))$.

It is customary in poverty measurement to define a poverty threshold identifying the poor and the non-poor populations (Sen, 1976). Suppose the social planner decides that category c_k for any $1 \leq k < S$ and $k \in \mathbb{N}$ is the *poverty threshold*, so that people experiencing categories $c_1, \ldots, c_k$ are identified as *poor*; whereas people experiencing categories $c_{k+1}, \ldots, c_S$ are identified as *non-poor*. We assume that at least one category reflects the absence of poverty, as this restriction is both intrinsically reasonable and required for stating certain properties in Section 2.1. When $k = 1$, only category c_1 reflects poverty and, in this case, $p_1(\mathbf{x})$ is the proportion of the population identified as poor. For any $c_k \in \mathbf{C}_{-S}$, we denote the *set of poor population* in $\mathbf{x}$ by $\mathbf{Z}^P(\mathbf{x}; \mathbf{C}, c_k) = \cup_{s=1}^{k} \Omega_s(\mathbf{x})$, the *set of non-poor population* by $\mathbf{Z}^{NP}(\mathbf{x}; \mathbf{C}, c_k) = \cup_{s=k+1}^{S} \Omega_s(\mathbf{x})$, and the *proportion of poor population* or the *headcount ratio* by $H(\mathbf{x};\mathbf{C},c_k) = \sum_{s=1}^{k} p_s(\mathbf{x})$.

We define a *poverty measure* $P(\mathbf{x};\mathbf{C},c_k)$ as $P : \mathbf{X} \times \mathbf{C} \times \mathbf{C}_{-S} \to \mathbb{R}_+$. In words, a poverty measure is a mapping from the set of actual individual deprivation vectors, the fixed set of categories, and the set of poverty thresholds to the real line. Especially, we propose the following class $\mathscr{P}$ of ordinal poverty measures, that are amicable to empirical applications and are policy-relevant:

$$P(\mathbf{x};\mathbf{C},c_k) = \sum_{s=1}^{S} p_s(\mathbf{x})\omega_s \tag{1}$$

where $\omega_1 = 1$, $\omega_{s-1} > \omega_s > 0$ for all $s = 2, ..., k$ whenever $k \geq 2$, and $\omega_s = 0$ for all $s > k$.

An ordinal poverty measure in our proposed class is a weighted sum of the population proportions in $\mathbf{p}(\mathbf{x})$, where the weights (i.e., ω_s's) are *non-negative* for all categories, *strictly positive* for the deprived categories, and *unity* for the most deprived category. We refer to weights ω_s's as *ordering weights* and to $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_S)$ as the *ordering weighting vector*.[5] The ordering weights increase with deprived categories representing higher levels of deprivation. In practice, the ordering weights may take various forms. For example, Bennett and Hatzimasoura (2011) make the ordering weight for each deprivation category depend on the latter's relative deprivation rank. Category s is assigned an ordering weight equal to $\omega_s = [(k - s + 1)/k]^\theta$ for all $s = 1, \ldots, k$ and for some $\theta > 0$. Thus, the least deprived category c_k receives an ordering weight of $\omega_k = 1/k^\theta$; whereas, the most deprived category c_1 receives an ordering weight of $\omega_1 = 1$.

The class of poverty measures in Equation 1 bears several policy-relevant features. First, unlike the headcount ratio, the poverty measures in our class are *sensitive to the depth of deprivations* as they assign larger weights to the more deprived categories. Thus, unlike the headcount ratio, the proposed measures are sensitive to changes in deprivation status among the poor even when they do not become non-poor owing to those changes. Second, the proposed poverty measures are *additively decomposable*, which has two crucial policy implications. One is that the society's overall poverty measure may be expressed as a population-weighted average of the population subgroup poverty measures, whenever the entire population is divided into mutually exclusive and collectively exhaustive population subgroups. The other is that additively decomposable measures are convenient for cross-sectional and inter-temporal econometric analysis as well as impact evaluation exercises. Third, the poverty measures are conveniently *normalised* between zero and one. They are equal to zero only in a society where nobody is poor; whereas, they are equal to one only whenever everybody in the society experiences the worst possible deprivation category c_1. Fourth, the poverty measure boils down to the *headcount ratio* either when the poverty threshold is represented by the most deprived category or whenever the underlying ordinal variable has only two categories.

2.1 Axiomatic Characterisation

We now provide an axiomatic characterisation of the class of measures proposed in Equation 1. In other words, certain reasonable and policy-relevant assumptions lead to this particular class of measures. Suppose the actual

[5] Kobus and Milos (2012, Theorem 3) also showed that a subgroup decomposable inequality measure for ordinal variables that is sensitive to spreads away from the median is a monotonic transformation of the *weighted sum of population proportions*. However, our ordering weights are significantly different both in terms of their restrictions and interpretation vis-a-vis those involved in ordinal inequality measurement. Moreover, our SCD property (see below) imposes a more stringent restriction on the permissible functional transformations.

deprivation level or poverty value of individual n in $\mathbf{x}$, experiencing category c_s for some $c_s \in \mathbf{C}$, is $x_n(c_s) = \omega_s(n,N) \in \mathbb{R}_+$. Two points should be noted in this general framework. One is that any two individuals experiencing the same deprivation category may not necessarily have the same levels of actual deprivations. The other is that the deprivation level of an individual may not be independent of the overall population size of the society in which they live. Now, the policymaker only observes the S deprivation categories experienced by individuals, not their actual deprivation levels. So, the policymaker needs to make certain assumptions to perform interpersonal comparisons and meaningfully aggregate the information available on the ordered deprivation categories in order to obtain a *cardinal poverty measure*. We present these assumptions in terms of properties or axioms. In fact, we state six properties that can be used to derive and axiomatically characterise the class of ordinal poverty measures ($\mathscr{P}$) proposed in Equation 1.

The first property is *anonymity*, which requires each person's identity to remain anonymous for the purpose of societal poverty evaluation. In other words, merely shuffling the individual deprivation levels among the people within a society, keeping population size unchanged, should not alter social poverty evaluation. This is an ethical principle from the policymaker's point of view. Technically, if a deprivation vector is post-multiplied by a *permutation matrix* to obtain another deprivation vector, then societal poverty should not change. For any $j \in \mathbb{N}\backslash\{1\}$, a *permutation matrix* $\mathscr{P}^j$ is a $j \times j$ non-negative square matrix with only one element in each row and each column being equal to one and the rest of the elements being equal to zero. For any two vectors $\mathbf{x},\mathbf{y} \in \mathbf{X}_N$, we say that $\mathbf{y}$ is obtained from $\mathbf{x}$ by *permutation* if $\mathbf{y} = \mathbf{x}\mathscr{P}^N$, where a permutation simply changes the position of the elements within a vector.

Anonymity (ANO) For any $\mathbf{x},\mathbf{y} \in \mathbf{X}_N$ and for any $c_k \in \mathbf{C}_{-S}$, if $\mathbf{y}$ is obtained from $\mathbf{x}$ by *permutation*, then $P(\mathbf{x};\mathbf{C},c_k) = P(\mathbf{y};\mathbf{C},c_k)$.

The second property is the *population principle*, which requires that a mere duplication of individual deprivation levels should not alter societal poverty evaluation. This property allows the social planner to compare societies with different population sizes as well as to compare societal poverty at different time periods. For any $\mathbf{x} \in \mathbf{X}_N$ and for any $\mathbf{y} \in \mathbf{X}_{N'}$, where $N' = r \times N$ for some $r \in \mathbb{N}\backslash\{1\}$, we say that $\mathbf{y}$ is obtained from $\mathbf{x}$ by *replication* whenever $\mathbf{y} = (\mathbf{x},\ldots,\mathbf{x})$. Note that a replication simply creates a multiplication of every element in deprivation vector $\mathbf{x}$ by $r > 1$ times to obtain another deprivation vector $\mathbf{y}$.

Population Principle (POP) For any $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and for any $c_k \in \mathbf{C}_{-S}$, if $\mathbf{y}$ is obtained from $\mathbf{x}$ by *replication*, then $P(\mathbf{x};\mathbf{C},c_k) = P(\mathbf{y};\mathbf{C},c_k)$.

The third property is *ordinal monotonicity*, which requires that if the deprivation level of a poor person improves so that the person moves to a lesser deprivation category, then societal poverty should be lower. The formal statement of the property requires that if a poor person moves from a category c_s reflecting poverty to a less deprived category $c_{s'}$, while the deprivation levels of every other person remain unchanged, then poverty should fall.

Ordinal Monotonicity (OMN) For any $\mathbf{x},\mathbf{y} \in \mathbf{X}_N$ and for any $c_k \in \mathbf{C}_{-S}$, if $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x};c_k)$ but $n' \in \Omega_{s'}(\mathbf{y})$ for any $s' > s$, while $x_n = y_n$ for all $n \neq n'$, then $P(\mathbf{y};\mathbf{C},c_k) < P(\mathbf{x};\mathbf{C},c_k)$.

The fourth property is *single-category deprivation*. The property requires that whenever there is only one category reflecting poverty (i.e. c_1), then the poverty measure should be equal to the headcount ratio $H(\cdot;\mathbf{C},c_1) = p_1(\cdot)$. In other words, we assume that whenever there is only one category reflecting poverty and the rest reflect an absence of poverty, then the headcount ratio becomes a sufficient statistic for the assessment of poverty. In fact, in this situation, any functional transformation of the headcount ratio would not add any meaningful information to the poverty assessment while being inferior in terms of intuitive interpretation.

Single-Category Deprivation (SCD) For any $\mathbf{x} \in \mathbf{X}$ and $c_1 \in \mathbf{C}_{-S}$, $P(\mathbf{x};\mathbf{C},c_1) = p_1(\mathbf{x})$.

The fifth property, *focus*, is essential for a poverty measure. It requires that, *ceteris paribus*, change in a non-poor person's situation should not alter societal poverty evaluation as long as non-poor person remains in that status. Given that the social planner cannot observe the actual deprivation levels of individuals, it is required that as long as the set of poor people remains unchanged within each of the k categories reflecting poverty, the level of poverty should be the same. Note that the set of non-poor people may remain unchanged or may be different across the $S-k$ categories *not* reflecting poverty, but this should not matter for poverty evaluation.

Focus (FOC) For any $\mathbf{x},\mathbf{y} \in \mathbf{X}_N$ and for any $c_k \in \mathbf{C}_{-S}$, if $\Omega_s(\mathbf{x}) = \Omega_s(\mathbf{y})\ \forall s \leq k$, then $P(\mathbf{x};\mathbf{C},c_k) = P(\mathbf{y};\mathbf{C},c_k)$.

Finally, the social planner may be interested in exploring the relationship between the overall poverty evaluation and the subgroup poverty evaluation, which requires some subgroup notation. Suppose the entire society with individual deprivation vector $\mathbf{x} \in \mathbf{X}_N$ is partitioned into $M \in \mathbb{N}\backslash\{1\}$ mutually exclusive and collectively exhaustive population subgroups, such that $\mathbf{x} = (\mathbf{x}^1,\ldots,\mathbf{x}^M)$. The individual deprivation vector of subgroup m is denoted by $\mathbf{x}^m \in \mathbf{X}_{N^m}$ for all m, where the population size of subgroup m is denoted by $N^m \in \mathbb{N}$, such that $\sum_{m=1}^{M} N^m = N$. The final property, *subgroup decomposability*, requires overall societal poverty to be expressible as a population-weighted average of subgroup poverty levels. This property has high policy relevance.

Subgroup Decomposability (SUD) For any $M \in \mathbb{N}/\{1\}$, for any $\mathbf{x} \in \mathbf{X}_N$ so that $\mathbf{x} = (\mathbf{x}^1,\ldots,\mathbf{x}^M)$ where $\mathbf{x}^m \in \mathbf{X}_{N^m}$ for some $N^m \in \mathbb{N}$ for all $m = 1,\ldots,M$, and for any $c_k \in \mathbf{C}_{-S}$,

$$P(\mathbf{x};\mathbf{C},c_k) = \sum_{m=1}^{M} \frac{N^m}{N} P(\mathbf{x}^m;\mathbf{C},c_k).$$

These six properties lead to the class $\mathscr{P}$ of poverty measures in Equation 1, which we present in Theorem 2.1:

Theorem 2.1 A poverty measure $P \in \mathscr{P}$ satisfies properties ANO, POP, OMN, SCD, FOC and SUD if and only if

$$P(\mathbf{x};\mathbf{C},c_k) = \sum_{s=1}^{S} p_s(\mathbf{x})\omega_s$$

where $\omega_1 = 1$, $\omega_{s-1} > \omega_s > 0$ for all $s = 2,\ldots,k$ whenever $k \geq 2$, and $\omega_s = 0$ for all $s > k$.

Proof. See Appendix A1. ■

A poverty measure satisfying these six stated properties turns out to be a weighted sum of population proportions. Note that, unlike most characterisations, the normalisation behaviour of our measures between zero and one is not an axiomatic assumption, but a logical conclusion from the foundational properties. Additionally, each poverty measure in class $\mathscr{P}$ may have the following alternative interpretation: when the policymaker is not able to observe people's actual deprivation levels but only observes the ordered deprivation categories that they experience, then, based on the six assumptions, the policymaker assigns particular deprivation values in the form of ω_s's to these individuals. The poverty measure is an average of these assigned deprivation values.

Even though additively decomposable measures are common in cardinal poverty measurement literature, our measures are reminiscent of the additively decomposable poverty measures within the multidimensional counting approach (Chakravarty and D'Ambrosio, 2006; Alkire and Foster, 2011; Bossert et al., 2013; Alkire and Foster, 2016; Dhongde et al., 2016), albeit with two subtle, yet crucial, differences. The first difference between our framework and the counting approach is conceptual. The counting framework starts by assuming the existence of binary deprivation values (usually "0" for non-deprivation and "1" for deprivation), which are then aggregated to obtain a *cardinal deprivation value* for each individual. These cardinal deprivation values are then aggregated to obtain an additively decomposable poverty measure. However, instead of directly cardinalising the level of deprivations, one may create ordered deprivation categories, such as 'deprived in all dimensions', 'deprived in some dimensions', 'deprived in one dimension', and so forth. If the probability masses in these categories are aggregated respecting the weights presented in Theorem 2.1, then a counting measures may be expressed as an ordinal poverty measure in our proposed class (discussed in detail in Section 6), but, in contrast to the counting approach, the use of weights as functions of cardinal deprivation values is not mandatory in our framework. The second difference is that our primary aim in this paper is to capture the depth of deprivation within a dimension. The measures proposed under the counting approach usually build from dichotomising deprivations for each dimension, thereby ignoring the depth of deprivations within dimensions.[6]

3 Precedence to the Poorer Among the Poor

Poverty alleviation is a gradual process, where it is imperative to ensure that the poorest of the poor are not *left behind*. Although all poverty measures in Equation (1) are sensitive to the depth of deprivations, not all of them ensure that the poorest among the poor population receive precedence over the less poor population during a poverty alleviation process. Giving precedence to those who are poorer has mostly been considered in the literature to be aligned with the *egalitarian view* of requiring poverty measures to be sensitive to the redistribution of achievements among the poor (Sen, 1976; Foster et al., 1984; Zheng, 1997). The egalitarian view, for cardinal variables, has been tantamount to requiring that a poverty measure should decrease due to rank-preserving *progressive transfers* and a poverty measure should increase due to *regressive transfers* among the poor. This is the famous Pigou-Dalton principle.

A rank-preserving progressive transfer takes place when some cardinally measurable *amount* of achievement of a *poor* person is transferred to a poorer person so that the transferee remains poorer than the transferor after the transfer. A regressive transfer takes place, on the other hand, when some cardinally measurable *amount* of achievement of a poor person is transferred to a better-off poor person. An essential rule of these particular transfers is that the gain or loss of the transferee is equal to the loss or gain of the transferor, respectively. Most cardinal poverty measures as well as many recently proposed counting poverty measures satisfy the Pigou-

[6]For a proposal to identify ultra-poor people according to the depth of their deprivations within the counting framework, see Alkire and Seth (2016). However, unlike our approach, Alkire and Seth (2016) did not develop a depth-sensitive poverty index.

Dalton principle.

In the absence of observable cardinally meaningful transferable achievements, however, it is hard to operationalise the traditional egalitarian concept in the ordinal framework. Yet the literature on distributional analysis with ordered categorical variables features two types of efforts to incorporate egalitarian concerns. One type of transfer, which is highly analogous to cardinal Pigou-Dalton transfer, entails a transfer of ranks between two persons (see, e.g. Bennett and Hatzimasoura, 2011) or a transfer of probability masses between two ordered categories (see, e.g. Silber and Yalonetzky, 2011; Yalonetzky, 2012). A second form of progressive transfer is the *Hammond transfer*, which captures the notion of inequality but without imposing the Pigou-Dalton restriction whereby the gain or loss of the transferee has to be equal to the loss or gain of the transferor (Hammond, 1976; Gravel et al., 2015). Though the concept of Hammond transfer has been applied in the context of inequality measurement and ordinal versions of the generalised Lorenz quasi-ordering, it has not yet been applied to poverty measurement.

The egalitarian view stems from the inequality measurement literature. However, there is a subtle, yet crucial, difference in the application of the egalitarian view to inequality measurement vis-a-vis poverty measurement. Unlike in inequality measurement, a progressive transfer and a regressive transfer may not be conceptually equivalent in the context of poverty measurement. In the case of progressive transfer between two poor persons, the set of poor population remains unchanged. However, in the case of a regressive transfer between two poor persons, the richer-poor person may either remain poor (i.e. keeping the set of the poor unchanged after the transfer) or become non-poor (i.e. the set of the poor changes after the transfer). Changes in the set of the poor, or lack thereof, may have different implications to poverty measurement. See, for instance, the relevant discussions in Foster and Sen (1997, pp. 174–175) regarding unidimensional cardinal poverty measures, and Datt (2018) in the context of counting measures.

The justification for implementing an egalitarian view in poverty measurement, especially the sensitivity to transfers between poor individuals, can be criticised, first, on the grounds that poverty alleviation exercises should chiefly be concerned with transfers of achievements (or resources) between the non-poor and the poor. From the perspective of a policymaker in charge of poverty alleviation, a transfer of achievements between two poor individuals does not appear to have much practical relevance. Second, giving precedence to poorer people should rather be aligned with the *prioritarian view*, which requires that 'benefiting people matters more the worse off these people are' (Parfit, 1997, p. 213).

Although, Fleurbaey (2015, p. 208) argues that 'a prioritarian will always find some egalitarians on her side', in this paper, we introduce an alternative, yet intuitive, concept of giving *precedence to poorer people* in the *ordinal* framework, more in tune with the prioritarian view.[7] Furthermore, we introduce a generalised framework to incorporate a form of *degree of precedence* to poorer people.[8]

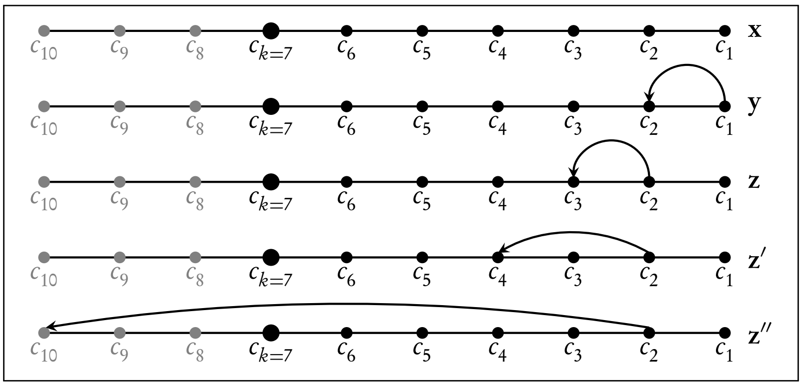
Let us introduce the concept using the example in Figure 1. Suppose there are ten ordered deprivation categories, where c_1 is the most deprived category and c_{10} is the least deprived category. Categories $c_1, \ldots, c_7$, denoted by black circles, reflect poverty; whereas, the other three categories, denoted by gray circles, do not reflect poverty. Thus, c_7 is the poverty threshold category, which is highlighted by a large black circle in each

[7] For an application of the prioritarian concept to the multidimensional context, see Bosmans et al. (2017).

[8] The concept is analogous to the degree of poverty or inequality aversion in the cardinal poverty measurement literature (Clark et al., 1981; Chakravarty, 1983; Foster et al., 1984), but not technically identical.

distribution. The original distribution is $\mathbf{x}$, where each individual experiences one of the ten categories.

Figure 1: Precedence to poorer and the degree of precedence



Suppose the policymaker has the following two competing options: either (a) obtain distribution $\mathbf{y}$ from $\mathbf{x}$ by assisting an individual to move from category c_1 to category c_2; or (b) obtain distribution $\mathbf{z}$ from $\mathbf{x}$ by assisting an individual to move from category c_2 to category c_3. Which option should lead to a larger reduction in poverty? One way of giving precedence to poorer people is to require that the move from $\mathbf{x}$ to $\mathbf{y}$ should lead to a larger reduction in poverty than the move from $\mathbf{x}$ to $\mathbf{z}$. It is a minimal criterion for giving precedence to poorer people which we refer to as *minimal precedence to poorer people* (PRE-M). This property requires that, *ceteris paribus*, moving a poorer person to an adjacent less deprived category leads to a larger reduction in poverty than moving a less poor person to a respectively adjacent less deprived category.[9] Following is a formal statement of the property:

Minimal Precedence to Poorer People (PRE-M) For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $k \geq 2$, for any $c_k \in \mathbf{C}_{-S}$, and for some $n' \neq n''$ and $s < t < S$ such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ and $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$, if (i) $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_{s+1}(\mathbf{y})$, while $x_n = y_n$ for all $n \neq n'$, and (ii) $\mathbf{z}$ is obtained from $\mathbf{x}$ such that $n'' \in \Omega_{t+1}(\mathbf{z})$, while $x_n = z_n$ for all $n \neq n''$, then $P(\mathbf{y}; \mathbf{C}, c_k) < P(\mathbf{z}; \mathbf{C}, c_k)$.

The PRE-M property presents a minimal criterion for giving precedence to poorer people. Yet what happens when the policymaker faces the alternatives of improving the situation of a poorer person by one category and improving the situation of a less poor person by several categories? To ensure that the policymaker still chooses to improve the situation of the poorer person in these cases, we introduce the property of *greatest precedence to poorer people* (PRE-G). This property requires that, *ceteris paribus*, moving a poorer person to a less deprived category leads to a larger reduction in poverty than moving a less poor person to *any number* of less deprived categories. Note here that the improvement is not restricted to a particular number of adjacent categories. For example, as in Figure 1, a move from $\mathbf{x}$ to $\mathbf{y}$, in this case, should lead to a larger reduction in poverty than even a move from $\mathbf{x}$ to $\mathbf{z}''$. The PRE-G property is stated as follows:

Greatest Precedence to Poorer People (PRE-G) For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $k \geq 2$, for any $c_k \in \mathbf{C}_{-S}$, and

[9] We have defined only the strict versions of these properties, requiring poverty to be strictly lower in the aftermath of specific pro-poorest distributional change. Consequently the ensuing results impose strict inequality restrictions on weights. However, these strict restrictions may be relaxed with alternative versions if the latter only require that poverty does not rise due to the same pro-poorest distributional change.

for some $n' \neq n''$ and $s < t < S$ such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ and $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$, if (i) $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_{s'}(\mathbf{y})$ for some $s < s' \leq t$, while $x_n = y_n$ for all $n \neq n'$, and (ii) $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_{t'}(\mathbf{z})$ for some $t' > t$, while $x_n = z_n$ for all $n \neq n''$, then $P(\mathbf{y}; \mathbf{C}, c_k) < P(\mathbf{z}; \mathbf{C}, c_k)$.

Conceptually, the PRE-G property is analogous to the notion of a Hammond transfer (Hammond, 1976; Gravel et al., 2015), which essentially involves, simultaneously, an improvement in a poor person's situation and a deterioration of a less poor person's situation, such that their deprivation ranks are not reversed (in the case of poverty measurement). Importantly, unlike the PRE-M property, the number of categories between s and s' does not need to be the same as the number of categories between t and t' in the case of PRE-G. An ordinal poverty measure satisfying property PRE-G also satisfies property PRE-M, but the reverse is not true. A policymaker supporting property PRE-G over property PRE-M should be considered more poverty averse.

We could also consider intermediate forms of preference between the minimal (PRE-M) and the greatest (PRE-G) forms of precedence. For example, instead of the greatest forms of precedence, the policymaker may be satisfied with requiring, that, *ceteris paribus*, moving a poorer person to an adjacent less deprived category leads to a larger reduction in poverty than moving a less poor person to, say, two adjacent less deprived categories. In terms of the diagram in Figure 1, in this case, a move from $\mathbf{x}$ to $\mathbf{y}$ should lead to a larger reduction in poverty than a move from $\mathbf{x}$ to $\mathbf{z}'$. This case may be referred to as *precedence to poorer people of order two* (PRE-2). In this way, we may obtain the policymaker's preferred degree α of giving precedence to poorer people. Thus, we introduce a general property referred to as *precedence to poorer people of order* α, which requires that, *ceteris paribus*, moving a poorer person to an adjacent less deprived category leads to a larger reduction in poverty than moving a less poor person up to an $\alpha\,(\geq 1)$ number of adjacent less deprived categories. A formal statement of the property is as follows:

Precedence to Poorer People of Order α (PRE-α) For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $k \geq 2$, for any $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, for any $c_k \in \mathbf{C}_{-S}$ and for some $n' \neq n''$ and $s < t < S$ such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ and $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$, if (i) $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_{s+1}(\mathbf{y})$, while $x_n = y_n$ for all $n \neq n'$, and (ii) $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_{t'}(\mathbf{z})$ for some $t' = \min\{t + \alpha, S\}$, while $x_n = z_n$ for all $n \neq n''$, then $P(\mathbf{y}; \mathbf{C}, c_k) < P(\mathbf{z}; \mathbf{C}, c_k)$.

Note that PRE-1 is essentially the PRE-M property. This is the case where the social planner is least poverty averse. As the value of α increases, the social planner's poverty aversion rises. In this framework, the social planner's poverty aversion is highest at $\alpha = k - 1$. We shall subsequently show that PRE-α for $\alpha = k - 1$ leads to the same subclass of ordinal poverty measures as the PRE-G property. The PRE-α property imposes further restrictions on the class of measures in Theorem 2.1. In Theorem 3.1, we present the subclass of measures $\mathscr{P}_\alpha$ that satisfy the general PRE-α property:

Theorem 3.1 For any $k \geq 2$ and for any $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, a poverty measure $P \in \mathscr{P}$ additionally satisfies property PRE-α if and only if:

a. $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+\alpha}$ $\forall s = 2, \ldots, k - \alpha$ and $\omega_{s-1} > 2\omega_s$ $\forall s = k - \alpha + 1, \ldots, k$ whenever $\alpha \leq k - 2$.
b. $\omega_{s-1} > 2\omega_s$ $\forall s = 2, \ldots, k$ whenever $\alpha = k - 1$.

Proof. See Appendix A2. ■

Theorem 3.1, in a novel effort, presents various subclasses of indices based on the degree of poverty aversion α, which we denote as $\mathscr{P}_\alpha$. In order to give precedence to poorer people, additional restrictions must be imposed on the ordering weights. Corollary 3.1 presents the special case of $\mathscr{P}_1$, featuring the least poverty averse social planner:

Corollary 3.1 For any $k \geq 2$, a poverty measure $P \in \mathscr{P}$ additionally satisfies property PRE-M (i.e. PRE-1) if and only if $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+1}$ for all $s = 2, ..., k-1$ and $\omega_{k-1} > 2\omega_k$.

Proof. The result follows directly from Theorem 3.1 by setting $\alpha = 1$. ■

To give precedence to poorer people in the spirit of property PRE-M, the ordering weights must be such that the difference $\omega_{s-1} - \omega_s$ is larger than the subsequent difference $\omega_s - \omega_{s+1}$, in addition to the restrictions imposed by Theorem 2.1. Suppose we summarise the ordering weights by: $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_S)$. Let us consider an example involving five categories and two ordering weight vectors: $\boldsymbol{\omega}' = (1, 0.8, 0.5, 0, 0)$ and $\boldsymbol{\omega}'' = (1, 0.5, 0.2, 0, 0)$, where $k = 3$. The ordering weights in $\boldsymbol{\omega}'$ fulfill all properties presented in Theorem 2.1, but the largest reduction in poverty is obtained whenever a poor person moves from the least poor category to the adjacent non-poor category. By contrast, ordering weights in $\boldsymbol{\omega}''$ require that the largest reduction in poverty be obtained whenever a poor person moves from the poorest category to the adjacent second poorest category. Thus, unlike the ordering weights in $\boldsymbol{\omega}'$, the ordering weights in $\boldsymbol{\omega}''$ make sure that poorer people receive precedence.

Next we present the subclass of poverty measures that satisfy property PRE-G, i.e. $\mathscr{P}_G$:

Proposition 3.1 For any $k \geq 2$, a poverty measure $P \in \mathscr{P}$ additionally satisfies property PRE-G if and only if $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$.

Proof. See Appendix A3. ■

The additional restriction on the ordering weights in Proposition 3.1 effectively prioritises the improvement in a poorer person's situation over improvement of any extent in a less poor person's situation. Let us consider an example involving five categories and two ordering weight vectors: $\boldsymbol{\omega}^1 = (1, 0.6, 0.3, 0.1, 0)$ and $\boldsymbol{\omega}^2 = (1, 0.48, 0.23, 0.1, 0)$, where $k = 4$. Clearly, both sets of weights in $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ satisfy the restriction in Corollary 3.1 that $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+1}$ for all $s = 2, \ldots, k$. However, the ordering weights in $\boldsymbol{\omega}^1$ do not satisfy the restriction in Proposition 3.1, since $\omega_1^1 < 2\omega_2^1$; whereas the ordering weights in $\boldsymbol{\omega}^2$ do satisfy the restriction in Proposition 3.1 as $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$.

An interesting feature of the set of weights satisfying property PRE-G is that, for $k \geq 3$, any deprivation category up to the third least detrimental deprivation category (i.e. $k-2$), should receive a weight greater than the sum of weights assigned to all categories reflecting lesser deprivation, i.e. $\omega_s > \sum_{\ell=s+1}^{k} \omega_\ell$ for all $s = 1, \ldots, k-2$.

Finally, it is worth pointing out that, remarkably, the subclasses $\mathscr{P}_S$ (Proposition 3.1) and $\mathscr{P}_{k-1}$ (Theorem 3.1 when $\alpha = k-1$) are identical – even though the distributional changes involved in the PRE-α property are only specific cases of the greatest precedence to poorer people involved in axiom PRE-G. Besides being of interest in itself, this perfect overlap between the subclasses of indices will prove useful in Section 5 because by deriving the dominance conditions for the subclasses $\mathscr{P}_\alpha$, we will also obtain the relevant dominance conditions for

Table 1: The five ordered categories of access to sanitation facilities

Category	Description
Open defecation	Human faeces disposed of in fields, forests, bushes, open bodies of water, beaches or other open spaces or disposed of with solid waste
Unimproved	Pit latrines without a slab or platform, hanging latrines and bucket latrines
Limited	Sanitation facilities of an otherwise acceptable type shared between two or more households
Basic unsafe	A basic sanitation facility which is not shared with other households, but excreta are not disposed safely, such as flushed but not disposed to piped sewer system, septic tank or pit latrine
Improved	Sanitation facility which is not shared with other households and where excreta are safely disposed in situ or treated off-site and includes flush/pour flush to piped sewer system, septic tank or pit latrine, ventilated improved pit latrine, composting toilet or pit latrine with a slab

subclass $\mathscr{P}_S$.[10]

4 Empirical Illustration: Sanitation Deprivation in Bangladesh

We now present an empirical illustration in order to showcase the efficacy of our proposed measurement method. In the current global development context, both the United Nations through the Sustainable Development Goals[11] and the World Bank through their Report of the Commission on Global Poverty (World Bank, 2017) have acknowledged the need for assessing, monitoring, and alleviating poverty in multiple dimensions besides the monetary dimension. In practice, most non-income dimensions are assessed by ordinal variables. In this section, we show how our measurement tools may be applied to analyse inter-temporal sanitation deprivation in Bangladesh.

For our analysis, we use the nationally representative Demographic Health Survey (DHS) datasets of Bangladesh for the years 2007, 2011, and 2014. While computing the estimates and the standard errors, we incorporate the sampling weights as well as respect the survey design.[12] Excluding the non-usual residents, we were able to use the information on 50,215 individuals from 10,398 households in the 2007 survey, 79,483 individuals from 17,139 households in the 2011 survey, and 77,680 individuals from 17,299 households in the 2014 survey.

One target of the United Nations' sixth Sustainable Development Goal (whose aim is to 'ensure availability and sustainable management of water and sanitation for all') is 'by 2030, [to] achieve access to adequate and equitable sanitation and hygiene for all and end open defecation.' In order to hit the target, the Joint Monitoring Programme (JMP) of the World Health Organisation and the UNICEF proposes using 'a *service ladder*

[10] Noteworthy is the expected resemblance between the weighting restriction identified in Proposition 3.1 and that in the class of welfare functions for ordinal variables derived in Gravel et al. (2015, Lemma 2). The latter characterises welfare functions that increase both when someone moves to a better category (so-called increments) and in the aftermath of Hammond transfers. Setting $\alpha_k = 0$ and changing the inequality sign in Gravel et al. (2015, Lemma 2) to interpret the α functions as ordering weights for poverty measurement yields the weight restriction in Proposition 3.1.

[11] Available at https://sustainabledevelopment.un.org/sdgs.

[12] See NIPORT et al. (2009, 2013, 2016) for details about the survey design.

approach to benchmark and track progress across countries at different stages of development", building on the existing datasets.[13] We pursue this service ladder approach and apply our ordinal poverty measures to study the improvement in sanitation deprivation in Bangladesh. We classify households' *access to sanitation* in the five ordered categories presented in Table 1. The five categories are ordered as 'open defecation' $\succ_D$ 'unimproved' $\succ_D$ 'limited' $\succ_D$ 'basic unsafe' $\succ_D$ 'improved'. We consider all persons living in a household deprived in access to sanitation if the household experiences any category other than the 'improved' category.

Table 2 shows how the estimated population shares in different deprivation categories have evolved over time in Bangladesh. Clearly, the estimated percentage in the 'improved' category has gradually increased (statistically significantly) from 28.5% in 2007 to 36.6% in 2011 to 47.8% in 2014. Thus, the proportion of the population in deprived categories has gone down over the same period. Changes within the deprived categories are however mixed. Although the estimated population shares in the two most deprived categories ('open defecation' and 'unimproved') have decreased (statistically significantly) systematically between 2007 and 2014, the population shares in the other two deprivation categories have not.

Has this estimated reduction pattern been replicated within all divisions? Table 2 also presents the changes in the discrete probability distributions in three divisions: Dhaka, Rajshahi, and Sylhet.[14] The estimated population shares in the 'improved' category have increased (statistically significantly) gradually in all three regions (Table 2), and so the shares of deprived population have gone down. We need, however, to point out two crucial aspects.

First, let us compare the reduction patterns in Dhaka and Rajshahi. The population share in the 'improved' category is higher in Rajshahi in 2011 and 2014 and statistically indistinguishable in 2007, implying that sanitation deprivation in Dhaka is never lower than sanitation deprivation in Rajshahi. However, the estimated population shares in the two most deprived categories ('open defecation' and 'unimproved') are higher in Rajshahi than in Dhaka in 2007 and 2014 and statistically indistinguishable in 2011. Second, like Dhaka and Rajshahi in Table 2, sanitation deprivation in Sylhet has also improved gradually. However, the estimated population share in the poorest category ('open defecation') is significantly higher in 2011 and in 2014 than in 2007. A simple headcount measure, which only captures the proportion of the overall deprived population, would always overlook these substantial differences.

Table 3 presents four different poverty measures for Bangladesh and for its six divisions. We assume the poverty threshold category to be 'basic unsafe'. The first poverty measure is the headcount ratio (H), which, in this context, is the population share experiencing any one of the four deprivation categories. The second measure is P_1, such that $P_1 \in \mathscr{P} \setminus \{\mathscr{P}_\alpha\}$ for $\alpha \geq 1$, and is defined by the ordering weights $\omega^1 = (1, 0.75, 0.5, 0.25, 0)$. The third measure is $P_2 \in \mathscr{P}_1 \setminus \{\mathscr{P}_\alpha\}$ for $\alpha \geq 2$ with ordering weights $\omega^2 = (1, 0.75^2, 0.5^2, 0.25^2, 0)$, i.e. respecting the restrictions in Corollary 3.1, but not respecting, for instance, the restrictions in Proposition 3.1 or the restrictions in Theorem 3.1 for $\alpha \geq 2$; whereas, the fourth measure is $P_3 \in \mathscr{P}_S$ with ordering weights $\omega^3 = (1, 0.4, 0.15, 0.05, 0)$, i.e. respecting the restrictions in Proposition 3.1. Note that measures P_2 and P_3 give precedence to those who are in the poorer categories. All four measures lie between zero and one, but we have multiplied them by one hundred so that they lie between zero (lowest deprivation) and 100 (highest

[13] The JMP document titled *WASH Post-2015: Proposed indicators for drinking water, sanitation and hygiene* was accessed in April 2017 at https://www.wssinfo.org.

[14] A new division called Rangpur was formed in 2010, which was a part of the Rajshahi Division. The Rangpur division did not exist during the 2007 DHS survey and so we have combined this new division with the Rajshahi division in the 2011 and 2014 DHS surveys to preserve comparability over time.

Table 2: Change in population distribution across sanitation categories in Bangladesh

	Bangladesh			Dhaka			Rajshahi			Sylhet		
	2007	2011	2014	2007	2011	2014	2007	2011	2014	2007	2011	2014
Open defecation	7.5	4.2	3.3	7.5	4.0	2.2	13.8	3.9	3.2	2.1	12.5	9.4
	(0.8)	(0.3)	(0.5)	(1.4)	(0.7)	(0.8)	(2.3)	(0.8)	(0.9)	(0.4)	(1.5)	(1.3)
Unimproved	47.1	38.3	25.7	44.3	35.9	22.6	45.3	36.4	28.3	57.2	34.3	23.0
	(1.1)	(0.9)	(1.3)	(1.9)	(1.7)	(2.8)	(2.6)	(3.1)	(2.7)	(3.3)	(2.0)	(2.6)
Limited	13.4	16.7	20.9	14.4	18.0	26.1	14.7	20.7	20.2	10.1	17.7	22.3
	(0.5)	(0.6)	(0.8)	(1.1)	(1.5)	(2.0)	(1.2)	(1.4)	(1.3)	(1.9)	(0.9)	(1.8)
Basic	3.5	4.3	2.3	8.6	10.5	5.4	0.2	0.2	0.3	0.6	0.1	0.3
	(0.4)	(0.5)	(0.4)	(1.0)	(1.6)	(1.0)	(0.1)	(0.1)	(0.1)	(0.2)	(0.1)	(0.2)
Not deprived	28.5	36.6	47.8	25.2	31.6	43.7	26.0	38.8	48.0	30.1	35.4	45.0
	(1.0)	(0.9)	(1.1)	(1.9)	(1.9)	(2.5)	(1.9)	(2.3)	(2.3)	(2.3)	(1.9)	(1.7)

Sources: Authors' own computations. Standard errors are reported in parentheses.

Table 3: Change in sanitation deprivation by ordinal poverty measures in Bangladesh and its divisions

	H			P_1			P_2			P_3		
	2007	2011	2014	2007	2011	2014	2007	2011	2014	2007	2011	2014
Barisal	66.1	60.5	46.8	47.7	44.0	32.6	35.0	32.6	23.4	25.2	23.5	16.7
	(2.7)	(2.1)	(3.3)	(1.9)	(1.7)	(2.9)	(1.4)	(1.4)	(2.4)	(1.1)	(1.1)	(1.8)
Chittagong	67.1	59.2	44.9	47.0	38.9	29.2	34.8	27.3	20.2	26.2	19.6	14.9
	(2.9)	(2.1)	(3.1)	(2.8)	(1.7)	(2.9)	(2.7)	(1.5)	(2.7)	(2.6)	(1.2)	(2.5)
Dhaka	74.8	68.4	56.3	50.1	42.6	33.5	36.6	29.4	21.8	27.8	21.6	15.4
	(1.9)	(1.9)	(2.5)	(1.7)	(1.3)	(1.8)	(1.6)	(1.1)	(1.6)	(1.4)	(1.0)	(1.3)
Khulna	69.0	61.4	50.3	48.9	41.8	33.0	35.7	29.4	22.7	25.9	20.9	16.1
	(1.8)	(1.6)	(2.3)	(1.4)	(1.2)	(1.7)	(1.1)	(1.0)	(1.4)	(1.0)	(0.7)	(1.0)
Rajshahi	74.0	61.2	52.0	55.2	41.6	34.7	43.0	29.6	24.2	34.1	21.6	17.6
	(1.9)	(2.3)	(2.3)	(1.8)	(1.9)	(1.8)	(1.9)	(1.6)	(1.5)	(1.9)	(1.3)	(1.3)
Sylhet	69.9	64.6	55.0	50.1	47.1	37.9	36.8	36.2	27.9	26.5	28.9	21.9
	(2.3)	(1.9)	(1.7)	(1.9)	(1.6)	(1.7)	(1.5)	(1.4)	(1.6)	(1.1)	(1.4)	(1.4)
Bangladesh	71.5	63.4	52.2	50.4	42.3	33.5	37.5	30.1	23.1	28.5	22.2	16.8
	(1.0)	(0.9)	(1.1)	(0.9)	(0.7)	(0.9)	(0.9)	(0.6)	(0.8)	(0.8)	(0.5)	(0.7)

Sources: Authors' own computations. Standard errors are reported in parentheses.

deprivation).

Comparisons of these measures provide useful insights, especially into the two crucial aspects that we have presented in Table 2. The headcount ratio estimate in Dhaka is statistically indistinguishable from the headcount ratio estimate in Rajshahi for 2007, despite deprivation in the two poorest categories being higher in Rajshahi. However, this crucial aspect is captured by the latter three measures, which show statistically significantly higher poverty estimates in Rajshahi than in Dhaka. Similarly, the headcount ratio estimate is higher in Dhaka than in Rajshahi for 2011, but the difference vanishes when poverty is assessed by the other three ordinal measures.

5 Poverty Dominance Conditions

Stochastic dominance conditions come in handy whenever we want to ascertain the robustness of a poverty ranking of distributions to alternative reasonable comparison criteria, e.g. selection of poverty lines, choice of different functional forms, etc. (Atkinson, 1987; Foster and Shorrocks, 1988b; Fields, 2001). Moreover, often stochastic dominance conditions reduce an intractable problem of probing the robustness of a comparison across an infinite domain of alternative criteria to a finite set of distributional tests (Levy, 2006). In Sections 2 and 3, we introduced the class of poverty measures $\mathscr{P}$ and its subclasses $\mathscr{P}_\alpha$ and $\mathscr{P}_S$. The main parameters for these measures are the set of ordering weights $\{\omega_1,\ldots,\omega_k\}$, the poverty threshold category c_k, and the poverty aversion parameter α. It is thus natural to inquiry into the circumstances under which ordinal poverty comparisons are robust to the alternative ordering weights as well as to the alternative poverty threshold categories. In this section, first we introduce the first-order dominance conditions relevant to $\mathscr{P}$, followed by the second-order dominance conditions for $\mathscr{P}_\alpha$ for all α.

In order to state the conditions we bring in some additional notation. First, we define the cumulative distribution function (CDF) of any distribution $\mathbf{x} \in \mathbf{X}$ as $F(\mathbf{x};\mathbf{C},c_s) \equiv \sum_{\ell=1}^{s} p_\ell(\mathbf{x})$ for all $s = 1,\ldots,S$. Clearly, $F(\mathbf{x};\mathbf{C},c_1) = p_1(\mathbf{x})$ and $F(\mathbf{x};\mathbf{C},c_S) = 1$. We denote the *difference operator* by Δ, and for any $\mathbf{x},\mathbf{y} \in \mathbf{X}$, express $\Delta P(\mathbf{x},\mathbf{y};\mathbf{C},c_k) \equiv P(\mathbf{x};\mathbf{C},c_k) - P(\mathbf{y};\mathbf{C},c_k)$ where $P(\cdot;\mathbf{C},c_k) = \sum_{s=1}^{S} p_s(\cdot)\omega_s$ from Equation (1); $\Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_s) \equiv F(\mathbf{x};\mathbf{C},c_s)$
$-F(\mathbf{y};\mathbf{C},c_s)$; and $\Delta p_s(\mathbf{x},\mathbf{y}) \equiv p_s(\mathbf{x}) - p_s(\mathbf{y})$. For notational convenience, we will often refer to $\Delta P(\mathbf{x},\mathbf{y};\mathbf{C},c_k)$ as ΔP_k, $\Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_s)$ as ΔF_s, and $\Delta p_s(\mathbf{x},\mathbf{y})$ as Δp_s.

5.1 First-order Dominance Conditions

Theorem 5.1 provides the first-order dominance conditions relevant to all measures in class $\mathscr{P}$ for a *given* poverty threshold category $c_k \in \mathbf{C}_{-S}$:

Theorem 5.1 For any $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and for any $k \geq 1$, $\Delta P(\mathbf{x},\mathbf{y};\mathbf{C},c_k) < 0$ for all $P \in \mathscr{P}$ for a given $c_k \in \mathbf{C}_{-S}$ if and only if $\Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_s) \leq 0$ for all $s \leq k$ with at least one strict inequality.

Proof. See Appendix A4. ∎

Theorem 5.1 states that poverty in one distribution $\mathbf{x} \in \mathbf{X}$ is strictly lower than in another distribution $\mathbf{y} \in \mathbf{X}$ for a chosen poverty threshold category $c_k \in \mathbf{C}_{-S}$ and for all measures $P \in \mathscr{P}$ if and only if the CDF of $\mathbf{x}$

is nowhere above and, at least once, is below the CDF of $\mathbf{y}$ up to category c_k. In other words, the poverty comparison for a particular poverty threshold category c_k is robust to all poverty measures $P \in \mathscr{P}$ if and only if $H(\mathbf{x};\mathbf{C},c_s) \leq H(\mathbf{y};\mathbf{C},c_s)$ for all $s \leq k$ and $H(\mathbf{x};\mathbf{C},c_s) < H(\mathbf{y};\mathbf{C},c_s)$ for at least one $s \leq k$.

Corollary 5.1 provides the first-order dominance condition relevant to any measure $P \in \mathscr{P}$ for *all* $c_k \in \mathbf{C}_{\_S}$:

Corollary 5.1 For any $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and for any $k \geq 1$, $\Delta P(\mathbf{x},\mathbf{y};\mathbf{C},c_k) < 0$ for any $P \in \mathscr{P}$ and for all $c_k \in \mathbf{C}_{\_S}$ if and only if $\Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_s) \leq 0$ for all $s = 2,\ldots,k$ and $\Delta p_1(\mathbf{x},\mathbf{y}) < 0$.

Proof. The sufficiency part is straightforward and follows from Equation A11. We prove the necessary condition as follows. First, consider $k = 1$. Then, $\Delta P_1 < 0$ only if $\Delta F_1 < 0$ or, equivalently, $\Delta p_1 < 0$. Subsequently, the requirement that $\Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_s) \leq 0$ for every $s = 2,\ldots,k$ follows from Theorem 5.1. ■

Interestingly, poverty in distribution $\mathbf{x}$ is lower than poverty in distribution $\mathbf{y}$ for any $P \in \mathscr{P}$ and for all possible poverty threshold categories if and only if $H(\mathbf{x};\mathbf{C},c_s) \leq H(\mathbf{y};\mathbf{C},c_s)$ for all $s \leq k$ and $H(\mathbf{x};\mathbf{C},c_1) < H(\mathbf{y};\mathbf{C},c_1)$. The results in Theorem 5.1 and Corollary 5.1 are the ordinal versions of the headcount-ratio orderings for continuous variables derived by Foster and Shorrocks (1988b).

5.2 Second-order Dominance Conditions

In Theorem 5.2 we first present the second-order general dominance conditions relevant to any measure in subclass $\mathscr{P}_\alpha$ for $\alpha \geq 1$ for a *given* poverty threshold category $c_k \in \mathbf{C}_{\_S}$ such that $k \geq 2$:

Theorem 5.2 For any $\mathbf{x},\mathbf{y} \in \mathbf{X}$, for any $k \geq 2$, and for any $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, $\Delta P(\mathbf{x},\mathbf{y};\mathbf{C},c_k) < 0$ for all $P \in \mathscr{P}_\alpha$ for a given $c_k \in \mathbf{C}_{\_S}$:

a. **if** $\sum_{\ell=1}^{s} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0$ for all $s = 1,\ldots,k$ with at least one strict inequality; and
b. **only if**:
   i. $\sum_{\ell=1}^{s} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0$ for all $s = 1,\ldots,k$ with at least one strict inequality, when $\alpha = 1$.
   ii. $\sum_{\ell=1}^{s} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0$ for all $s = 1,\ldots,k-\alpha+1$ and $(\sum_{\ell=1}^{k-\alpha} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell)) + (\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell)) + 2^{1-\alpha} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_k) \leq 0$ with at least one strict inequality; when $2 \leq \alpha \leq k-1$.

Proof. See Appendix A5. ■

Thus, poverty in $\mathbf{x}$ is lower than poverty in $\mathbf{y}$ for a given poverty threshold category, for all $P \in \mathscr{P}_\alpha$, and for a given α if all cumulations of the cumulative distribution of $\mathbf{x}$ are nowhere above and, at least once, are strictly below that of $\mathbf{y}$ up to the poverty threshold category c_k. Note that this second-order sufficiency requirement is the same no matter the value of α. However, the sufficient condition is not necessary. The necessary condition depends on the value of α. It is only for $\alpha = 1$, i.e. the case of $P \in \mathscr{P}_1$ satisfying property PRE-M, that the necessary condition is identical to the sufficient condition. In fact, the condition for $\alpha = 1$ is the ordinal version of the 'P_2' poverty ordering as in Foster and Shorrocks (1988b).

Unlike the case of $\alpha = 1$, the necessary conditions diverge from the sufficient conditions whenever $\alpha \geq 2$. The number of restrictions to check for necessity decreases as the value of α increases. For $\alpha = k-1$, only three such restrictions must be checked. Corollary 5.2 presents the necessary condition for $\Delta P_k < 0$ when $\alpha = k-1$ or whenever the poverty measures satisfy property PRE-G:

Corollary 5.2 For any $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and for any $k \geq 2$, $\Delta P(\mathbf{x},\mathbf{y};\mathbf{C},c_k) < 0$ for all $P \in \mathscr{P}_S$ for a given $c_k \in \mathbf{C}_{-S}$ only if $\sum_{\ell=1}^{s} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0$ for $s = 1,2$ and $\Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_1) + (\sum_{\ell=2}^{k-1} 2^{1-\ell} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell)) + 2^{2-k} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_k) \leq 0$ with at least one strict inequality.

Proof. It is straightforward to verify from Theorem 5.2 by setting $\alpha = k - 1$. ■

Finally, Corollary 5.3 provides the second-order dominance conditions relevant to any measure $P \in \mathscr{P}_\alpha$ for *all* $c_k \in \mathbf{C}_{-S}$ such that $k \geq 2$:

Corollary 5.3 For any $\mathbf{x},\mathbf{y} \in \mathbf{X}$, $\Delta P(\mathbf{x},\mathbf{y};\mathbf{C},c_k) < 0$ for any $P \in \mathscr{P}_\alpha$ and for all $c_k \in \mathbf{C}_{-S} \setminus \{c_1\}$

a. **if** *either* $[(\sum_{\ell=1}^{2} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0) \wedge (\Delta p_1(\mathbf{x},\mathbf{y}) < 0)]$ *or* $[(\sum_{\ell=1}^{2} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) < 0) \wedge (\Delta p_1(\mathbf{x},\mathbf{y}) \leq 0)]$ whenever $k = 2$ and additionally $\sum_{\ell=1}^{s} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0 \; \forall s = 3,\ldots,k$ whenever $k \geq 3$; and

b. **only if** either $[(\sum_{\ell=1}^{2} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0) \wedge (\Delta p_1(\mathbf{x},\mathbf{y}) < 0)]$ or $[(\sum_{\ell=1}^{2} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) < 0) \wedge (\Delta p_1(\mathbf{x},\mathbf{y}) \leq 0)]$ whenever $k = 2$ and additionally $\sum_{\ell=1}^{s} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell) \leq 0$ for all $s = 3,\ldots,k-\alpha+1$ and $(\sum_{\ell=1}^{k-\alpha} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell)) + (\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_\ell)) + 2^{1-\alpha} \Delta F(\mathbf{x},\mathbf{y};\mathbf{C},c_k) \leq 0$ whenever $k \geq 3$.

Proof. First consider the case when $k = 2$. From Equation A12, we obtain $\Delta P_2 = (\omega_1 - 2\omega_2)\Delta F_1 + \omega_2(\Delta F_1 + \Delta F_2)$. It is easy to verify that in order to have $\Delta P_2 < 0$, it is both necessary and sufficient that either $(\sum_{\ell=1}^{2} \Delta F_\ell \leq 0 \wedge \Delta p_1 < 0)$ or $(\sum_{\ell=1}^{2} \Delta F_\ell < 0 \wedge \Delta p_1 \leq 0)$. The additional sufficient and necessary condition for $k \geq 3$ follows from Theorem 5.2. ■

In summary, the robustness of poverty comparisons for various classes and subclasses of ordinal measures introduced in Sections 2.1 and 3 can be assessed with a battery of dominance tests based on the theorems and corollaries presented in this section.

6 Application to Multidimensional Poverty Measurement

So far we have focused on a single dimension. However, the literature on multidimensional poverty has grown significantly over the last two decades and so has the surrounding debate. Several multidimensional poverty measures have been proposed in the literature since the seminal work of Chakravarty et al. (1998) and Bourguignon and Chakravarty (2003) under the assumption that the underlying dimensions are cardinal. Yet one important concern has been how to conduct meaningful poverty assessment when the underlying dimensions are ordinal. This challenge has not been overlooked and various multidimensional poverty measures motivated by the *counting approach* (Atkinson, 2003) have been proposed (see, for instance, Chakravarty and D'Ambrosio, 2006; Alkire and Foster, 2011; Aaberge and Peluso, 2012; Bossert et al., 2013; Alkire and Foster, 2016; Dhongde et al., 2016). We have already mentioned that our approach to measure poverty with ordinal variables is motivationally and conceptually different from the counting framework, and yet their functional forms are very similar. In this section, we flesh out this similarity followed by a discussion of how our approach to giving *precedence to poorer* may be extended beyond the existing counting approach framework.

Multidimensional counting measures are based on simultaneous deprivations across different dimensions. Formally, additively decomposable counting poverty measures are constructed in the following steps for a hypothetical society with N individuals and $\mathscr{D} \geq 2$ dimensions. First, if an individual n is deprived in dimension

d, then the person is assigned a *deprivation status score* of $g_{nd} = 1$; whereas, the person is assigned a score of $g_{nd} = 0$, otherwise. The same goes for all $n = 1, \ldots, N$ and for all $d = 1, \ldots, \mathscr{D}$.

Second, a relative weight w_d is assigned to the d^{th} deprivation, such that $w_d > 0$ and $\sum_{d=1}^{\mathscr{D}} w_d = 1$, and an *attainment score* $\sigma_n = \sum_{d=1}^{\mathscr{D}} w_d(1 - g_{nd})$ is obtained for all $n = 1, \ldots, N$.[15] By construction, $0 \leq \sigma_n \leq 1$ for all n and a larger attainment score reflects a lower level of deprivation. Let us denote the n attainment scores by $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$. Given that $\mathscr{D}$ is finite, each weighting choice generates a finite number of S attainment scores, in turn creating S categories $c_1, \ldots, c_S$, such that category c_S reflects the lowest level of multiple deprivations and category c_1 reflects the largest level of multiple deprivations. We denote the attainment score corresponding to category c_s by $\mathscr{C}_s$ so that whenever individual n experiences category c_s, $\sigma_n = \mathscr{C}_s$. We denote the proportion of population experiencing score $\mathscr{C}_s$ by $p_s(\boldsymbol{\sigma})$.

In the third step, a category c_k for any $k < S$ is selected as a poverty threshold category to identify the poor, such that all people experiencing category c_s for all $s \leq k$ are identified as poor. The additively decomposable counting measures (P^C) are expressed as

$$P^C(\boldsymbol{\sigma}; \mathbf{C}, c_k) = \sum_{s=1}^{S} f(\mathscr{C}_s) p_s(\boldsymbol{\sigma}); \tag{2}$$

where $f(\mathscr{C}_1) = 1$, $f(\mathscr{C}_s)$ is monotonically decreasing in its argument for all $s \leq k$, and $f(\mathscr{C}_s) = 0$ for all $s > k$.

Different measures use different functional forms of $f(\mathscr{C}_s)$. For example, Alkire and Foster (2011) use $f(\mathscr{C}_s) = (1 - \mathscr{C}_s)$ for all $s \leq k$; Chakravarty and D'Ambrosio (2006) propose, inter alia, $f(\mathscr{C}_s) = (1 - \mathscr{C}_s)^\beta$ for $\beta \geq 1$ and for all $s \leq k = S - 1$; whereas Alkire and Foster (2016) use $f(\mathscr{C}_s) = (1 - \mathscr{C}_s)^\alpha$ for $\alpha \geq 1$ and for all $s \leq k$.[16] Note that Equations (1) and (2) are identical to each other, as the restrictions on $f(\mathscr{C}_s)$ are the same as the restrictions on ω_s for all s. Thus, the additively decomposable multidimensional counting measures can be expressed as the class of ordinal poverty measures in Theorem 2.1.

One controversial aspect surrounding the counting measures presented in Equation (2) is that they require assigning a precise weight (i.e. w_d) to each dimensional deprivation (Ravallion, 2011; Ferreira and Lugo, 2013). If there is agreement about a set of precise weights, then indeed counting approaches are highly amicable to policy applications.[17] However, if there is a unanimous agreement only about the ordinal ranking of different combinations of deprivations, but dissent about the precise weights, then is a non-counting multidimensional approach feasible using the ordinal measurement method that we have developed?

A second source of controversy is that counting approaches for ordinal variables are not developed for capturing the depth of deprivations within dimensions because each dimension is first dichotomised into sets of deprived and non-deprived people. Yet capturing the depth of deprivations within dimensions may be of great policy interest.[18] Can a non-counting multidimensional approach be applied using our ordinal measurement method?

Our answer to both questions is *yes*, which can be demonstrated with an example. Suppose poverty is assessed by using two dimensions: E and H, where there are two categories (E_1, E_2) in dimension E and three categories

[15] Note that the relative weights w_d's assigned to dimensions are different from the ordering weights ω_s's.

[16] Alkire and Foster (2016) use the parameter values of $\alpha \geq 0$, but we ignore the value of $\alpha = 0$ as the measure becomes the multidimensional headcount ratio.

[17] For discussions on weights in the counting approach framework, see Alkire et al. (2015, Chapter 6).

[18] For an approach to identification (but not aggregation) using a depth approach versus using an intensity approach in a counting framework, see Alkire and Seth (2016).

(H_1, H_2, H_3) in dimension H. Category E_1 in dimension E and categories H_1 and H_2 in dimension H reflect deprivations. Suppose we have the following information about the ordering of deprivations: (i) $E_1 \succ_D E_2$, (ii) $H_1 \succ_D H_2 \succ_D H_3$, and (iii) $H_2 \succ_D E_1$. The first two conditions present the ordering within each dimension; whereas the third conveys that a single deprivation in any category of dimension H is worse than a single deprivation in dimension E. Here, we are only aware of the ordinal ranking of deprivation categories within as well as across two dimensions.

In the multidimensional context, poverty is a reflection of different combinations of deprivations in different dimensions obtained through an identification function. The three aforementioned restrictions, along with the additional restriction that multiple deprivations are worse than a single deprivation, leads to the following ranking among the poor:

$$(E_1, H_1) \succ_D (E_1, H_2) \succ_D (E_2, H_1) \succ_D (E_2, H_2) \succ_D (E_1, H_3) \succ_D (E_2, H_3).$$

Thus, there are now six ordered categories (i.e. $S = 6$). Under the assumption that the poverty threshold category is $c_k = (E_2, H_2)$, a person must be deprived in dimension H in order to be identified as poor, leaving four ordered poverty categories ($k = 4$). Clearly, Theorem 2.1 as well as the concept of *precedence to poorer people* are applicable to this case. This example may be easily extended to cases involving more than two dimensions.

Notice that we neither considered any precise set of weights nor dichotomised all dimensions. Instead, dimension H had more than one deprivation category. Hence, not only do we argue that the multidimensional counting approaches can be expressed as the ordinal poverty measures in Theorem 2.1, but also that our ordinal poverty measures may be used for a broader, more holistic multidimensional framework.

7 Concluding Remarks

There is little doubt that poverty is a multidimensional concept and the current global development agenda correctly seeks to 'reduce poverty in all its dimensions'. To meet this target, it is indeed important to assess poverty from a multidimensional perspective. However, one should not discount the potential interest in evaluating the impact of a targeted program in reducing deprivation in a single dimension such as educational or health outcomes and access to public services, which may often be assessed by an ordinal variable with multiple ordered deprivation categories. The frequently used headcount ratio, in this case, is ineffective as it overlooks the depth of deprivations, i.e., any changes within the ordered deprivation categories.

Our paper has thus posed the question: 'How should we assess poverty when variables are ordinal?' Implicitly, the companion question is 'Can we meaningfully assess poverty beyond the headcount ratio when we have an ordinal variable?' Drawing on six reasonable axiomatic properties, our answer is 'Poverty can be measured with ordinal variables through weighted averages of the discrete probabilities corresponding to the ordered categories.' We refer to these weights as ordering weights, which need to satisfy a specific set of restrictions in order to ensure the social poverty indices fulfil these key properties. Our axiomatically characterised class of social poverty indices has certain desirable features, such as additive decomposability and being bounded between zero (when none experiences any deprivation) and one (when everyone experiences the most deprived category).

In contrast to previous attempts in the literature on poverty measurement with ordinal variables, we have gone fruitfully further in the direction of operationalising different concepts of 'precedence to the poorer people among the poor', which ensures that the policymaker has an incentive to assist the poorer over the less poor. We have shown that it is possible to devise reasonable poverty measures that prioritise welfare improvements among the most deprived when variables are ordinal. We have axiomatically characterised a set of subclasses of ordinal poverty measures based on a continuum of different notions of precedence to the poorer among the poor. Each subclass is defined by an additional restriction on the admissible ordering weights. The precedence-sensitive measures have proven useful in the illustration pertaining to sanitation deprivation in Bangladesh by highlighting those provinces where the overall headcount improvement did not come about through reductions in the proportion of the population in the most deprived categories.

Since several poverty measures are admissible within each characterised class and subclass, we have also developed stochastic dominance conditions for each subclass of poverty measures. Their fulfilment guarantees that all measures within a given class (or subclass) rank the same pair of distributions robustly. While some of these conditions represent the ordinal-variable analog of existing conditions for continuous variables in the poverty dominance literature (Foster and Shorrocks, 1988b), others are, to the best of our knowledge, themselves a novel methodological contribution to the literature on stochastic dominance with ordinal variables.

Considering the recent surge in the literature on multidimensional poverty measurement, especially on the counting approach, we showed how our method is closely aligned with the aggregation procedure characteristic of the counting framework. It is still the usual practice to dichotomise deprivations within each dimension when using existing counting measures, ignoring the depth within each category. Future research could focus on how to further develop the counting measures in order to incorporate the depth of deprivations in an ordinal framework.

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Appendices

Appendix A1 Proof of Theorem 2.1

It is straightforward to check that each poverty measure in Equation 1 satisfies the six properties: ANO, POP, OMN, SCD, FOC and SUD. We now prove the necessary part that if a poverty measure satisfies these six properties, then it takes the functional form in Equation 1.

By property SUD, for $M \in \mathbb{N}\backslash\{1\}$ mutually exclusive and collectively exhaustive population subgroups, such that $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^M)$, we already know that

$$P(\mathbf{x};\mathbf{C},c_k) = \sum_{m=1}^{M} \frac{N^m}{N} P(\mathbf{x}^m;\mathbf{C},c_k), \tag{A1}$$

where $N^m \in \mathbb{N}$ is the population size of subgroup m for all $m = 1, \dots, M$, such that $\sum_{m=1}^{M} N^m = N$.

By repeated application of SUD to Equation A1 until $M = N$, we obtain

$$P(\mathbf{x};\mathbf{C},c_k) = \frac{1}{N}\sum_{n=1}^{N} x_n(c_s) = \frac{1}{N}\sum_{s=1}^{S}\sum_{n=1}^{N_s(\mathbf{x})} \omega_s(n,N). \tag{A2}$$

Now suppose the deprivation vector $\mathbf{x} \in \mathbf{X}_N$ is such that $n \in \Omega_s(\mathbf{x})$ for all $n \in \mathbf{N}(\mathbf{x})$ and for some $c_s \in \mathbf{C}$. Suppose further that there is some $n' \in \mathbf{N}(\mathbf{x})$ such that $\omega_s(n',N) = \bar{\omega}'$ and $\omega_s(n,N) = \bar{\omega} \neq \bar{\omega}'$ for all $n \neq n'$. Next, let deprivation vector $\mathbf{y} \in \mathbf{X}_N$ be obtained from $\mathbf{x}$ by permutation so that there is some $n'' \in \mathbf{N}(\mathbf{y})$ such that $\omega_s(n'',N) = \bar{\omega}'$ and $\omega_s(n,N) = \bar{\omega}$ for all $n \neq n''$. In this case, property ANO requires that $P(\mathbf{x};\mathbf{C},c_k) = P(\mathbf{y};\mathbf{C},c_k)$ and clearly $N[P(\mathbf{x};\mathbf{C},c_k) - P(\mathbf{y};\mathbf{C},c_k)] = \omega_s(n',N) - \omega_s(n'',N) = 0$. Given that we choose any two arbitrary n' and n'' and $\mathbf{N}(\mathbf{x}) = \mathbf{N}(\mathbf{y})$, this leads to $\omega_s(n,N) = \omega_s(N)$ for all $n \in \Omega_s(\mathbf{x})$. Moreover, given that we choose an arbitrary $c_s \in \mathbf{C}$, the result holds for all $c_s \in \mathbf{C}$. Thus, Equation A2 may be expressed as

$$P(\mathbf{x};\mathbf{C},c_k) = \frac{1}{N}\sum_{s=1}^{S}\sum_{n=1}^{N_s(\mathbf{x})} \omega_s(N) = \sum_{s=1}^{S}\frac{N_s(\mathbf{x})}{N}\omega_s(N) = \sum_{s=1}^{S} p_s(\mathbf{x})\omega_s(N). \tag{A3}$$

Next, suppose again that $n \in \Omega_s(\mathbf{x})$ for all $n \in \mathbf{N}(\mathbf{x})$ for some $\mathbf{x} \in \mathbf{X}_N$ and for some $c_s \in \mathbf{C}$. Clearly, $P(\mathbf{x};\mathbf{C},c_k) = \omega_s(N)$ from Equation A3, since $p_s(\mathbf{x}) = 1$. Now, suppose deprivation vector $\mathbf{z} \in \mathbf{X}_{N'}$ is obtained from $\mathbf{x}$ by replication, such that $N' = \gamma N$ for some $\gamma \in \mathbb{N} \setminus \{1\}$. Then, $P(\mathbf{z};\mathbf{C},c_k) = \omega_s(N')$ because, again, $p_s(\mathbf{z}) = 1$. By property POP, we have $P(\mathbf{x};\mathbf{C},c_k) = P(\mathbf{z};\mathbf{C},c_k)$. It then follows that $\omega_s(N) = \omega_s(N')$. Since the relationship holds for two arbitrary population sizes N and N', it follows that $\omega_s(N) = \omega_s$ for all $N \in \mathbb{N}$ and for every $c_s \in \mathbf{C}$. Equation A3 thus may be rewritten as

$$P(\mathbf{x};\mathbf{C},c_k) = \sum_{s=1}^{S} p_s(\mathbf{x})\omega_s. \tag{A4}$$

Now, consider any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$, such that $n \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x};c_k)$ for all $n \in \mathbf{N}(\mathbf{x})$, but $n \in \Omega_{s'}(\mathbf{y})$ for all $n \in$

$\mathbf{N}(\mathbf{y})$ for some $s' > s$. Note in this case that $p_s(\mathbf{x}) = p_{s'}(\mathbf{y}) = 1$. Then, from property OMN, we know that $P(\mathbf{y};\mathbf{C},c_k) < P(\mathbf{x};\mathbf{C},c_k)$. Therefore, using Equation A4, we obtain

$$\omega_{s'} < \omega_s. \tag{A5}$$

The relationship in Equation A5 holds for any s, such that $s \leq k$ but $s < s'$. In other words, $\omega_{s-1} > \omega_s > \omega_{s'}$ for all $s = 2,...,k$ and for any $s' > k$, whenever $k \geq 2$. When $S = 2$, then $k = 1$ and so $\omega_1 > \omega_2$.

We next use property SCD. Suppose $k = 1$. Then, property SCD leads to $P(\mathbf{x};\mathbf{C},c_k) = p_1(\mathbf{x})$ and Equation A4 yields

$$P(\mathbf{x};\mathbf{C},c_k) = \sum_{s=1}^{S} p_s(\mathbf{x})\omega_s = p_1(\mathbf{x}). \tag{A6}$$

Note, by definition, that $0 \leq p_1(\mathbf{x}) \leq 1$ and so $0 \leq P(\mathbf{x};\mathbf{C},c_k) \leq 1$. Consider the situation where $n \in \Omega_1(\mathbf{x})$ for all $n \in \mathbf{N}(\mathbf{x})$. In this case, $p_1(\mathbf{x}) = 1$ and $p_s(\mathbf{x}) = 0$ for all $s \neq 1$. Clearly, from Equation A6, $\omega_1 = 1$. Moreover, from Equation A5, it follows that $\omega_{s-1} > \omega_s > \omega_{s'}$ for all $s = 2,\ldots,k$ and for any $s' > k$ whenever $k \geq 2$.

In order to complete the proof, we need to show that $\omega_s = 0$ for all $s > k$. For this purpose, consider any $\mathbf{x},\mathbf{y} \in \mathbf{X}_N$, such that $n \in \Omega_{s'}(\mathbf{x}) \subseteq \mathbf{Z}^{NP}(\mathbf{x};c_k)$ for all $n \in \mathbf{N}(\mathbf{x})$ and $n \in \Omega_{s''}(\mathbf{y}) \in \mathbf{Z}^{NP}(\mathbf{y};c_k)$ for all $n \in \mathbf{N}(\mathbf{y})$ for any $s'' > s' > k$. Note that $p_{s'}(\mathbf{x}) = p_{s''}(\mathbf{y}) = 1$ and indeed $p_s(\mathbf{x}) = p_s(\mathbf{y}) = 0 \ \forall s \leq k$. By property FOC, we then require $P(\mathbf{y};\mathbf{C},c_k) = P(\mathbf{x};\mathbf{C},c_k)$. Thus, from Equation A4, we obtain $P(\mathbf{x};\mathbf{C},c_k) = \omega_s = \omega_{s'} = P(\mathbf{y};\mathbf{C},c_k)$ for any $s' > s > k$. Since, $p_s(\mathbf{x}) = p_s(\mathbf{y}) = 0 \ \forall s \leq k$, it follows that $p_1(\mathbf{x}) = p_1(\mathbf{y}) = 0$. Consider $k = 1$. Then, by property SCD, we must have $P(\mathbf{x};\mathbf{C},c_k) = P(\mathbf{y};\mathbf{C},c_k) = 0$. Hence, it must be the case that $\omega_s = 0$ for all $s > k$, which completes our proof. ■

Appendix A2 Proof of Theorem 3.1

The sufficiency part is straightforward. We prove the necessity part as follows.

Suppose $k \geq 2$ and $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$. Now, suppose $\mathbf{y}$ and $\mathbf{z}$ are obtained from $\mathbf{x} \in \mathbf{X}_N$ as follows: For some $n'' \neq n'$ and some $t > s'$, $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_{s'}(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x};c_k)$ but $n' \in \Omega_{s'+1}(\mathbf{y})$, while $x_n = y_n$ for all $n \neq n'$; whereas, $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x};c_k)$ but $n'' \in \Omega_{t'}(\mathbf{z})$ for some $t > s'$ and $t' = \min\{t+\alpha, S\}$, while $x_n = z_n$ for all $n \neq n''$. It follows that $\Omega_s(\mathbf{y}) = \Omega_s(\mathbf{x})$ for all $s \neq s', s'+1$ and $\Omega_s(\mathbf{z}) = \Omega_s(\mathbf{x})$ for all $s \neq t, t'$; whereas $\Omega_{s'}(\mathbf{y}) = \Omega_{s'}(\mathbf{x}) - 1$, $\Omega_{s'+1}(\mathbf{y}) = \Omega_{s'+1}(\mathbf{x}) + 1$, $\Omega_t(\mathbf{z}) = \Omega_t(\mathbf{x}) - 1$, and $\Omega_{t'}(\mathbf{z}) = \Omega_{t'}(\mathbf{x}) + 1$. Note that by construction $t \leq k$.

By property PRE-α, we know that

$$P(\mathbf{y};\mathbf{C},c_k) - P(\mathbf{z};\mathbf{C},c_k) < 0. \tag{A7}$$

Combining Equations 1 and A7, we get

$$\omega_{s'+1} - \omega_{s'} - \omega_{t'} + \omega_t < 0.$$

Substituting $t = s' + 1 = s$ for any $s = 2, \ldots, k$, we obtain

$$\omega_{s-1} - \omega_s > \omega_s - \omega_{t'}. \tag{A8}$$

First, suppose $t' = s + \alpha \leq k < S$ or $s \leq k - \alpha$. Then $\omega_{t'} = \omega_{s+\alpha} > 0$ by Theorem 2.1 and Equation A8 can be expressed as $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+\alpha}$ for all $s = 2, \ldots, k - \alpha$. Second, suppose $t' = \min\{s + \alpha, S\} > k$ or $s > k - \alpha$. We know that $\omega_s = 0$ for all $s > k$ by Theorem 2.1 and so Equation A8 can be expressed as $\omega_{s-1} - \omega_s > \omega_s$ or $\omega_{s-1} > 2\omega_s$ for all $s = k - \alpha + 1, \ldots, k$. This completes the proof. ■

Appendix A3 Proof of Proposition 3.1

Let us first prove the sufficiency part. Suppose $k \geq 2$. We already know from Theorem 2.1 that $\omega_{s-1} > \omega_s > 0$ for all $s = 2, \ldots, k$ and $\omega_s = 0$ for all $s > k$. Suppose additionally that $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$. Alternatively, $\omega_{s-1} - \omega_s > \omega_s$ for all $s = 2, \ldots, k$.

For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $c_k \in \mathbf{C}_{-S}$, and for some $n'' \neq n'$, suppose $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_v(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n' \in \Omega_{v+\alpha}(\mathbf{y})$ for some $\alpha \in \mathbb{N}$, while $x_n = y_n$ for all $n \neq n'$, and $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ for some $t \geq v + \alpha$ but $n'' \in \Omega_{t+\beta}(\mathbf{z})$ for some $\beta \in \mathbb{N}$, while $x_n = z_n$ for all $n \neq n''$. By definition, $t \leq k$.

It follows that $\Omega_s(\mathbf{y}) = \Omega_s(\mathbf{x})$ for all $s \neq v, v+\alpha$ and $\Omega_s(\mathbf{z}) = \Omega_s(\mathbf{x})$ for all $s \neq t, t+\beta$; whereas $\Omega_v(\mathbf{y}) = \Omega_v(\mathbf{x}) - 1$, $\Omega_{v+\alpha}(\mathbf{y}) = \Omega_{v+\alpha}(\mathbf{x}) + 1$, $\Omega_t(\mathbf{z}) = \Omega_t(\mathbf{x}) - 1$, and $\Omega_{t+\beta}(\mathbf{z}) = \Omega_{t+\beta}(\mathbf{x}) + 1$. With the help of Equation 1, we get

$$P(\mathbf{y}; \mathbf{C}, c_k) - P(\mathbf{z}; \mathbf{C}, c_k) = \omega_{v+\alpha} - \omega_v - \omega_{t+\beta} + \omega_t = \left(\omega_t - \omega_{t+\beta}\right) - \left(\omega_v - \omega_{v+\alpha}\right). \tag{A9}$$

By assumption of the sufficiency part: $\omega_{s-1} - \omega_s > \omega_s$ for all $s = 2, \ldots, k$. Combining this assumption with the weight restrictions from Theorem 2.1 we can easily deduce that $(\omega_v - \omega_{v+\alpha}) > (\omega_v - \omega_{v+1}) > \omega_{v+1} > \omega_{v+\alpha}$. Hence, $(\omega_v - \omega_{v+\alpha}) > \omega_{v+\alpha}$. Since $v + \alpha \leq t \leq k$ and $\omega_{s-1} > \omega_s > 0$ for all $s = 2, \ldots, k$, it also follows that $\omega_{v+\alpha} \geq \left(\omega_t - \omega_{t+\beta}\right)$. Hence, $\left(\omega_v - \omega_{v+\alpha}\right) > \left(\omega_t - \omega_{t+\beta}\right)$ and $P(\mathbf{y}; \mathbf{C}, c_k) < P(\mathbf{z}; \mathbf{C}, c_k)$.

We next prove the necessity part starting with Equation A9. By property PRE-G, we know that $P(\mathbf{y}; \mathbf{C}, c_k) < P(\mathbf{z}; \mathbf{C}, c_k)$. Thus,

$$\omega_v - \omega_{v+\alpha} > \omega_t - \omega_{t+\beta}. \tag{A10}$$

Now the inequality in Equation A10 must hold for any situation in which $t \geq v + \alpha$, including the comparison of the minimum possible improvement for the poorer person, given by $\omega_v - \omega_{v+1}$ (i.e. with $\alpha = 1$), against the maximum possible improvement for the less poor person, given by $\omega_t - \omega_{t+\beta}$ with $t = v + 1$ and $t + \beta > k$. Inserting these values into Equation A10, bearing in mind that $\omega_{t+\beta} = 0$ when $t + \beta > k$, yields

$$\omega_v - \omega_{v+1} > \omega_{v+1}.$$

Substituting $v = s - 1$ for any $s = 2, \ldots, k$ yields $\omega_{s-1} - \omega_s > \omega_s$. Hence, $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$. ■

Appendix A4 Proof of Theorem 5.1

We first prove the sufficiency part. From Theorem 2.1, we know that $\omega_s = 0$ for all $s > k$. Thus, Equation 1 may be presented using the difference operator as $\Delta P_k = \sum_{s=1}^{k} \omega_s \Delta p_s$. Using summation by parts, also known as Abel's lemma (Guenther and Lee, 1988), it follows that

$$\Delta P_k = \sum_{s=1}^{k-1} [\omega_s - \omega_{s+1}] \Delta F_s + \Delta F_k \omega_k. \tag{A11}$$

We already know from Theorem 2.1 that $\omega_k > 0$ and $\omega_s - \omega_{s+1} > 0 \ \forall s = 1, \ldots, k-1$. Therefore, clearly from Equation A11, the condition that $\Delta F_s \leq 0$ for all $s \leq k$ and $\Delta F_s < 0$ for at least one $s \leq k$ is sufficient to ensure that $\Delta P_k < 0$ for all $P \in \mathscr{P}$ and for a given $c_k \in \mathbf{C}_{-\mathrm{S}}$.

We next prove the necessity part by contradiction. Consider the situation, where $\Delta F_t > 0$ for some $t \leq k$, $\Delta F_s \leq 0$ for all $s \leq k$ but $s \neq t$, and $\Delta F_s < 0$ for some $s \leq k$ but $s \neq t$. For a sufficiently large value of $\omega_t - \omega_{t+1}$ in Equation A11, it may always be possible that $\Delta P_k > 0$. Or, consider the situation $\Delta F_s = 0$ for all $s \leq k$. In this case, $\Delta P_k = 0$. Hence, the necessary condition requires both $\Delta F_s \leq 0$ for all $s \leq k$ and $\Delta F_s < 0$ for some $s \leq k$. This completes the proof. ■

Appendix A5 Proof of Theorem 5.2

Summing by parts the first component on the right-hand side of Equation A11 yields

$$\Delta P_k = \sum_{s=1}^{k-2} \left(\{[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}]\} \sum_{\ell=1}^{s} \Delta F_\ell \right) + [\omega_{k-1} - \omega_k] \sum_{s=1}^{k-1} \Delta F_s + \Delta F_k \omega_k. \tag{A12}$$

a. *Sufficiency*: Define $\lambda_s(\alpha) = (\omega_{s-1} - 2\omega_s + \omega_{s+\alpha}) + (\omega_{s+1} - \omega_{s+\alpha})$ for all $s = 2, \ldots, k-\alpha$ and $\eta_s(\alpha) = (\omega_{s-1} - 2\omega_s) + \omega_{s+1}$ for all $s = k-\alpha+1, \ldots, k-1$. Then the first component in Equation A12 can be decomposed into two components and the last two components may be rearranged to rewrite the equation as

$$\begin{aligned} \Delta P_k = & \sum_{s=2}^{k-\alpha} \left(\lambda_s(\alpha) \sum_{\ell=1}^{s-1} \Delta F_\ell \right) + \sum_{s=k-\alpha+1}^{k-1} \left(\eta_s(\alpha) \sum_{\ell=1}^{s-1} \Delta F_\ell \right) \\ & + (\omega_{k-1} - 2\omega_k) \sum_{s=1}^{k-1} \Delta F_s + \omega_k \sum_{s=1}^{k} \Delta F_s. \end{aligned} \tag{A13}$$

Given that $\omega_{s-1} - 2\omega_s + \omega_{s+\alpha} > 0$ for all $s = 2, \ldots, k-\alpha$ by Theorem 3.1 and $\omega_{s+1} - \omega_{s+\alpha} \geq 0$ for any $1 \leq \alpha \leq k-1$ by Theorem 2.1, we know that $\lambda_s(\alpha) > 0$ for all $s = 2, \ldots, k-\alpha$. Similarly, given that $\omega_{s-1} - 2\omega_s > 0$ for all $s = k-\alpha+1, \ldots, k-1$ by Theorem 3.1 and $\omega_{s+1} \geq 0$ by Theorem 1, we know that $\eta_s(\alpha) > 0$ for all $s = k-\alpha+1, \ldots, k-1$. We further know that $\omega_{k-1} - 2\omega_k > 0$ by Theorem 3.1 and that $\omega_k > 0$ by Theorem 2.1. It is now straightforward to check from Equation A13 that $\sum_{\ell=1}^{s} \Delta F_\ell \leq 0 \ \forall s \leq k$ and $\sum_{\ell=1}^{s} \Delta F_\ell < 0$ for at least one $s \leq k$, suffice for $\Delta P_k < 0$ for a given $c_k \in \mathbf{C}_{-\mathrm{S}}$.

b.i. *Necessity when* $\alpha = 1$: We can rewrite equation A12 the following way:

$$\Delta P_k = \sum_{s=1}^{k-1}\left(\{[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}]\}\sum_{\ell=1}^{s}\Delta F_\ell\right) + \omega_k \sum_{s=1}^{k}\Delta F_s. \tag{A14}$$

We know from Theorem 3.1 that $[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}] > 0 \ \forall s = 1, ..., k-1$. Likewise $\omega_k > 0$. Yet we do not have any further restriction stating whether any of the weight functions in Equation A14 are strictly greater than the others. Therefore every sum of cumulatives ($\sum_{\ell=1}^{s}\Delta F_\ell,\ s = 1, ..., k$) must be non-positive and at least one must be strictly negative in order to ensure $\Delta P_k < 0$.

b.ii. *Necessity when* $2 \le \alpha \le k-1$: Define additionally $\delta_s(\alpha) = (\omega_{s-1} - 2\omega_s)$, such that $\delta_s(\alpha) = \eta_s(\alpha) - \omega_{s+1}$, for all $s = k-\alpha+1, \dots, k-1$. Then, the middle two components of Equation A13 may be rearranged to rewrite the equation as

$$\begin{aligned}\Delta P_k = &\sum_{s=2}^{k-\alpha}\left(\lambda_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \sum_{s=k-\alpha+1}^{k-\alpha+2}\left(\delta_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right)\\ &+ \sum_{s=k-\alpha+3}^{k}\left(\delta_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \sum_{s=k-\alpha+1}^{k}\left(\omega_{s+1}\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \omega_k\sum_{s=1}^{k}\Delta F_s.\end{aligned} \tag{A15}$$

We already know that $\lambda_s(\alpha) > 0$ for all $s = 2, \dots, k-\alpha$ by Theorem 3.1 and Theorem 2.1 and it turns out that any of these weights can be larger than the other components' weights, it is thus necessary to have $\sum_{\ell=1}^{s}\Delta F_\ell \le 0$ for all $s = 1, \dots, k-\alpha-1$. Next, we also know that $\delta_s(\alpha) > 0$ for all $s = k-\alpha+1, \dots, k-1$. However, not all $\delta_s(\alpha)$'s are necessarily larger than other component weights. Whenever $k-\alpha+3 \le k$, it turns out that $\omega_{s-1} > \delta_s = \omega_{s-1} - 2\omega_s$ for all $s = k-\alpha+2, \dots, k$. Therefore, even when the third component in Equation A15 is positive, it is possible to have $\Delta P_k < 0$ whenever the final two components are negative and carry sufficiently larger relative weight, given that the first two components are not positive. Hence, it is necessary that the final three components are jointly non-positive, provided the largest possible weights are assigned to the final two components, which requires $\omega_s \to \omega_{s-1}/2$ for all $s = k-\alpha+3, \dots, k$. Consequently, $\delta_s \to 0$ for all $s = k-\alpha+3, \dots, k$. We use these conditions in the final three components of Equation A15 to obtain

$$\begin{aligned}\Delta P_k = &\sum_{s=2}^{k-\alpha}\left(\lambda_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \sum_{s=k-\alpha+1}^{k-\alpha+3}\left(\delta_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right)\\ &\omega_{k-\alpha+2}\left[\left(\sum_{\ell=1}^{k-\alpha}\Delta F_\ell\right) + \left(\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell}\Delta F_\ell\right) + 2^{1-\alpha}\Delta F_k\right].\end{aligned} \tag{A16}$$

Given that $\omega_{k-\alpha+2}$ may be higher than the weights of the rest of the components, it is necessary that the third component be not positive. Hence, it is necessary that $\sum_{\ell=1}^{s}\Delta F_\ell \le 0$ for all $s \le k-\alpha+1$ and $(\sum_{\ell=1}^{k-\alpha}\Delta F_\ell) + (\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell}\Delta F_\ell) + 2^{1-\alpha}\Delta F_k \le 0$ with at least one strict inequality for having $\Delta P_k < 0$. This completes the proof. ■