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Dimensional and Distributional Contributions to Multidimensional Poverty

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Abstract

The adjusted headcount ratio M_0 of Alkire and Foster (2011a) is increasingly being adopted by countries and international organizations to measure poverty. Three properties are largely responsible for its growing use: *Subgroup Decomposability*, by which an assessment of subgroup contributions to overall poverty can be made, facilitating regional analysis and targeting; *Dimensional Breakdown*, by which an assessment of dimensional contributions to overall poverty can be made after the poor have been identified, facilitating coordination; and *Ordinality*, which ensures that the method can be used in cases where variables only have ordinal meaning. Following Sen (1976), a natural question to ask is whether sensitivity to inequality among the poor can be incorporated into this multidimensional framework. We propose a *Dimensional Transfer* axiom that applies to multidimensional poverty measures and specifies conditions under which poverty must fall as inequality among the poor decreases. An intuitive transformation is defined to obtain multidimensional measures with desired properties from unidimensional FGT measures having analogous properties; in particular, Dimensional Transfer follows from the standard Transfer axiom for unidimensional measures. A version of the unidimensional measures yields the *M-gamma* class M_0^γ containing the multidimensional headcount ratio for $\gamma = 0$, the adjusted headcount ratio M_0 for $\gamma = 1$, and a *squared count* measure for $\gamma = 2$, satisfying Dimensional Transfer. Other examples show the ease with which measures can be constructed that satisfy Subgroup Decomposability, Ordinality, and Dimensional Transfer. However, none of these examples satisfies Dimensional Breakdown. A general impossibility theorem explains why this is so: Dimensional Breakdown is effectively inconsistent with Dimensional Transfer. Given the importance of Dimensional Breakdown for policy analysis, we suggest maintaining the adjusted headcount ratio as a central measure, augmented by the squared count measure or other indices that capture inequality among the poor. The methods are illustrated with an example from Cameroon.

Keywords: poverty measurement, multidimensional poverty, inequality, transfer axiom, dimensional breakdown, FGT measures, decomposability, axioms, identification, ordinal variables.

JEL classification: I3, I32, D63, O1, H1

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1. Introduction

The multidimensional poverty measures of Alkire and Foster (2011a) were designed with practical applications in mind, and there are many examples that illustrate their usefulness in monitoring poverty, targeting poor populations, and coordinating poverty reduction efforts in real world settings.¹ The effectiveness of these measures originates in the properties they satisfy, and these include: *Subgroup Decomposability*, by which an assessment of subgroup contributions to overall poverty can be made, thus facilitating regional analysis and targeting, and *Dimensional Breakdown*, by which an assessment of dimensional contributions to overall poverty can be made, thereby facilitating policy coordination.² In addition, the most common measure – the adjusted headcount ratio – satisfies a third property that ensures its broad applicability: *Ordinality* allows it to be used in the all too common cases where variables only have ordinal significance.

A natural question to ask is whether inequality can be usefully incorporated into this form of poverty measurement. Following Sen (1976), the literature on income (or unidimensional) poverty expresses the concern for inequality using a transfer principle that requires poverty to fall as result of a progressive transfer among the poor. This in turn has led to an array of distribution-sensitive income poverty measures satisfying this property.³ In the multidimensional setting, there are two competing notions of inequality, leading to two distinct ways of conceiving of inequality in multidimensional poverty. The first, linked most closely to Kolm (1977), generalizes the notion of a progressive transfer (or more broadly a Lorenz comparison) to the multidimensional setting by applying the same bistochastic matrix to every variable.⁴ This results in a coordinated “smoothing” of the distributions that preserves their means. The associated transfer principle for poverty measures requires poverty to

¹ See Atkinson (2003) concerning the practicality of counting measures. For a global application see Alkire and Santos (2010, 2014); official national poverty applications include Castillo Añezco and Jácome Pérez (2016) for Ecuador, CONPES (2012) for Colombia, Gob. De Chile (2015) for Chile, Royal Govt. of Bhutan (2010, 2012) for Bhutan; other applications include the Gross National Happiness Index of Bhutan (Ura *et al.* 2012, 2015) and the Women’s Empowerment in Agriculture Index (Alkire *et al.* 2013).

² Subgroup Decomposability (or Decomposability) is defined for unidimensional measures in Foster, Greer, and Thorbecke (1984) and for multidimensional measures in Chakravarty, Mukherjee, and Renade (1998), Tsui (2002), and Bourguignon and Chakravarty (2003). Dimensional Breakdown and Ordinality were outlined in Alkire and Foster (2011a) and are formally presented below.

³ See Sen (1976), Clark, Hemming, and Ulph (1981), and Foster, Greer, and Thorbecke (1984) among others. It should be noted that a property properly depends on both the identification and aggregation steps. In unidimensional measurement, identification usually has a standard format, so we often say that the poverty measure satisfies a given property without explicitly specifying the identification method.

⁴ A bistochastic matrix is a weighted average of different permutation matrices (each of which switches achievements around). When applied to an income distribution it ensures that each person’s transformed income is a weighted average of all the original incomes. See Foster and Sen (1997) or Alkire *et al.* (2015).

fall, or at least not to rise, when such a smoothing is applied among the poor. The second form of multidimensional inequality is linked to the work of Atkinson and Bourguignon (1982) and relies on patterns of achievements *across* dimensions. Imagine a case where one person initially has more of everything than another person and the two persons switch achievements in a single dimension. This can be interpreted as a progressive transfer that preserves the marginal distribution of each dimensional variable and lowers inequality by relaxing the positive association across variables. The resulting transfer principle specifies conditions under which this form of progressive transfer among the poor should lower poverty, or at least not raise it.

Many multidimensional poverty methodologies satisfy one or both of these transfer principles.⁵ In particular, it was shown in Alkire and Foster (2011a) that the adjusted Foster-Greer-Thorbecke (FGT) measures M_α , when used with a dual-cutoff method of identification, satisfy the first type of transfer principle for $\alpha \geq 1$ and the second type for $\alpha \geq 0$. Note, though, that the transfer properties in the multidimensional poverty literature are “weak” in that they allow poverty to remain unchanged in the face of a progressive transfer.⁶ In particular, the adjusted poverty gap measure M_1 , which is thoroughly insensitive to either form of transfer, satisfies both. It is possible to define associated strict versions of the properties that require poverty to fall as a result of a suitably strict progressive transfer, and to show that for $\alpha > 1$ the measure M_α satisfies a strict version of the first transfer principle, while for $\alpha > 0$ it is easily transformed into a *new* measure satisfying a strict version of the second (as outlined in the paper).⁷ However, each of these distribution-sensitive measures violates Ordinality, thus severely limiting its applicability. This leads to the following natural questions: Is it possible to formulate a strict version of distribution sensitivity – by which greater inequality among the poor strictly raises poverty – that is applicable to multidimensional poverty methodologies satisfying Ordinality? And can we find measures satisfying this requirement and also Subgroup Decomposability and Dimensional Breakdown, which have proved to be so useful in practice?

This paper considers the possibility of constructing multidimensional poverty measures satisfying these three properties and a strict form of distribution sensitivity called *Dimensional Transfer*. This

⁵ See for example Chakravarty, Mukherjee, and Renade (1998), Tsui (2002), Bourguignon and Chakravarty (2003), Chakravarty and D’Ambrosio (2006), Maasoumi and Lugo (2008), and Bossert, Chakravarty, and D’Ambrosio (2013).

⁶ See, for example, the related axioms of Tsui (2002), Chakravarty (2009), Bourguignon and Chakravarty (2003), and Alkire and Foster (2011a). For a clarifying discussion of this issue, see Alkire *et al.* (2015). Rippin (2013) has unfortunately falsely claimed that M_0 violates a property she calls “Sensitivity to Inequality Increasing Switch” which also has only weak inequalities. This and other inaccuracies in Table 1.01 of Rippin (2013) continue to be repeated; see for example Rippin (2015).

⁷ See Alkire and Foster (2011a, p. 485), where it is noted that one could replace the individual poverty function $M_\alpha(y_i; z)$ with $[M_\alpha(y_i; z)]^\gamma$ for some $\gamma > 0$ and average across persons.

property is based on an Atkinson-Bourguignon form of multidimensional transfer, but with the additional proviso that the participants are both poor and that the poorer person is deprived in the switched dimensions while the other poor person is not – so that it is also a transfer of a *deprivation* from a worse off poor person to a better off poor person. Following Alkire and Foster (2011a) we expand the adjusted headcount ratio M_0 to a parametric class M_0^γ – called here the *M-gamma* class – having a subclass that satisfies Dimensional Transfer. We then present an intuitive procedure for transforming *any* unidimensional poverty measure into a multidimensional poverty measure by constructing the *attainment count distribution* and applying the unidimensional poverty measure.⁸ Examples produced by the transformation include the global *Multidimensional Poverty Index* (or MPI), the official Columbian MPI, Mexico’s multidimensional methodology, and the *M-gamma* class.⁹ The transformation also effectively converts properties for unidimensional poverty measures into the corresponding properties for multidimensional measures. For instance, *Monotonicity* of the income poverty measure ensures that the resulting multidimensional poverty measure satisfies *Dimensional Monotonicity* as defined in Alkire and Foster (2011a). Subgroup Decomposability for multidimensional measures is likewise obtained from the associated property for unidimensional measures. To construct multidimensional measures satisfying Dimensional Transfer, it turns out that any unidimensional measure satisfying the standard Transfer principle will do. Hence it is straightforward to construct any number of multidimensional poverty measures that satisfy the usual properties: Ordinality, Subgroup Decomposability, and Dimensional Transfer. Any standard decomposable, unidimensional poverty measure satisfying the Transfer principle will do.

Unfortunately, it is *also* true that every one of these examples violates Dimensional Breakdown, a property that has proved especially crucial for conducting policy analysis. We identify the reasons for the violation and prove an impossibility result that essentially demonstrates the mutual incompatibility of Dimensional Breakdown and Dimensional Transfer. Consequently, Dimensional Transfer carries with it an opportunity cost: it is not freely available for measures satisfying Subgroup Decomposability, Dimensional Breakdown, and Ordinality. Given the key role of these properties for multidimensional poverty measurement, we recommend using methods satisfying the three, augmented by measures or information that separately accounts for inequality among the poor. In particular, the adjusted headcount ratio, which itself is neutral with respect to the transfers in the

⁸ A person’s *deprivation score* is the sum of the values of deprivations experienced by the person; the *attainment count* is the sum of the values associated with the remaining dimensions – those in which the person is not deprived. See section 4.

⁹ See Alkire and Santos (2010, 2014), CONPES (2012), and CONEVAL (2009), respectively. The *M-gamma* class is generated by versions of the FGT or *P-alpha* measures.

Dimensional Transfer property, can be used with other M -gamma measures in a manner reminiscent to the traditional P -alpha or FGT measures. The methods are illustrated with an example from Cameroon.

The basic definitions and notation used in this paper are given in Section 2, while Section 3 describes the key properties for multidimensional poverty measures. Section 4 presents the M -gamma class and describes the method of constructing multidimensional measures from unidimensional measures. General propositions linking unidimensional and multidimensional properties are provided, which, among other things, shows how to construct multidimensional measures satisfying the Dimensional Transfer property. Section 5 presents the impossibility result and outlines some potential ways forward with the help of an example, while Section 6 concludes.

2. Notation and Definitions

We begin with notation and definitions that will be needed in the subsequent analysis. Let $|v|$ denote the sum of all elements in any given vector or matrix v of real numbers, and let $\mu(v)$ signify the mean of v , or $|v|$ divided by the total number of elements in v . Where v and v' are vectors having the same number of entries, let $v > v'$ denote the case where v vector dominates v' (so that each coordinate of v is as large as the respective coordinate of v' , while $v \neq v'$).

In what follows, we consider allocations of dimensional achievements across populations. The number of dimensions is assumed to be a fixed integer $d \geq 2$, where the typical dimension is $j = 1, 2, \dots, d$. The population size is any integer $n \geq 1$, where n is permitted to range across the positive integers, and $i = 1, 2, \dots, n$ denotes the typical person. Let $y = [y_{ij}]$ be an $n \cdot d$ matrix of achievements belonging to the domain $Y = \{y \in R_+^{nd} : n \geq 1\}$ of nonnegative real matrices.¹⁰ The typical entry in y is $y_{ij} \geq 0$. We use y_i to signify the row vector of individual i 's achievements, while y_j is the column vector that provides the distribution of dimension j 's achievements across people. A *deprivation cutoff* $z_j > 0$ for dimension j is compared to achievement level y_{ij} to determine when person i is deprived in j , namely, when $y_{ij} < z_j$. The row vector of dimension-specific deprivation cutoffs is denoted by z .

Poverty measurement has an identification step and an aggregation step. An *identification function* $\rho: R_+^d \cdot R_+^d \rightarrow \{0, 1\}$ is used to identify whether person i is poor, where $\rho(y_i; z)$ takes the value of one if person i is poor, and the value of zero otherwise, and is weakly decreasing in each y_{ij} (lowering

¹⁰ We follow Alkire and Foster (2011a) in assuming that achievements are represented as nonnegative real numbers, while deprivation cutoffs are strictly positive. Other assumptions are clearly possible but are not explicitly covered here.

achievements does not bring a poor person out of poverty).¹¹ The *identification vector* associated with y is the column vector r whose i^{th} entry is $\rho(y_i; z)$, while the set of persons who are identified by ρ as being poor is denoted by $Z \subseteq \{1, \dots, n\}$. An *index or measure of multidimensional poverty* $M: Y \cdot R_{++}^d \rightarrow R_+$ aggregates the data into an overall level $M(y; z)$ of poverty in y given z and the identification function ρ . The resulting *methodology* for measuring multidimensional poverty is given by $\mathcal{M} = (\rho, M)$. For any given dimension j , let Y_j be the set of all column vectors y_j of j^{th} dimensional achievements. It will sometimes be useful to focus on y and y_j that are consistent with a given poverty status vector r . Let Y_r denote the set of y having r as their poverty status vector, and let Y_{rj} denote the set of all y_j that are derived from an achievement matrix y found in Y_r .

Alkire and Foster (2011a,b) identify and measure poverty using a vector of deprivation values and an overall poverty cutoff. Let $w_j > 0$ denote the *weight or deprivation value* of j and let w be the row vector satisfying $|w| = d$. The *poverty cutoff* is denoted by k , where $0 < k \leq d$. For any person i , the *deprivation score (or count)* c_i is the sum of the deprivation values w_j across all dimensions in which i is deprived. The *dual-cutoff identification function* ρ_k is defined by $\rho_k(y_i; z) = 1$ whenever $c_i \geq k$, and $\rho_k(y_i; z) = 0$ whenever $c_i < k$. In other words, ρ_k identifies person i as poor when the deprivation count c_i is at least k and i is not poor otherwise.¹² Let $Z(k)$ be the set of persons who are identified by ρ_k as being poor. At one extreme, when $k = d$, the function ρ_k becomes *intersection* identification, in which a person must be deprived in all dimensions to be poor. When $0 < k \leq \min_j w_j$, it becomes *union* identification, in which a person need be deprived in only one dimension to be identified as poor. Thus, while our emphasis is on the intermediate cases, ρ_k includes the two limiting identification methods.

Using deprivation cutoffs and values, we convert the matrix of achievements into a matrix focusing on deprivations. Let $g^0 = [g_{ij}^0]$ denote the deprivation matrix whose typical element is given by $g_{ij}^0 = w_j$ when $y_{ij} < z_j$, and $g_{ij}^0 = 0$ otherwise. In words, when person i is deprived in the j^{th} dimension, the associated entry g_{ij}^0 is the deprivation value w_j ; otherwise it is 0. Column vector g_j^0 clearly indicates those who are deprived in dimension j , while the i^{th} row vector of g^0 is person i 's *deprivation vector*, denoted g_i^0 . Summing the values in g_i^0 yields the deprivation count $c_i = |g_i^0|$ as defined above, while further dividing by the maximal value d yields the *deprivation share or score* $s_i = c_i/d$, from which the

¹¹ While not included in Alkire and Foster (2011a), this requirement would seem to be a reasonable restriction on ρ and the orientation of achievements.

¹² As noted below it is also possible to use $c_i > k$ in the definition of the poor.

column vector s of deprivation scores is constructed. It lists the intensity (or breadth) of deprivation experienced by each person.

The poverty cutoff k can be used to create a matrix focused on the deprivations of the poor. Let $g^0(k)$ denote the censored deprivation matrix whose typical element is given by $g_{ij}^0(k) = g_{ij}^0 \rho_k(y_i; z)$, which leaves the entries of the poor unchanged, while changing those of the nonpoor to zero.¹³ The column vector $g_{\cdot j}^0(k)$ contains w_j for every person who is both poor and deprived in dimension j , and 0 otherwise. Person i 's censored deprivation vector $g_i^0(k)$ is the i^{th} row of $g^0(k)$. The censored vector of deprivation scores $s(k)$, given by $s_i(k) = |g_i^0(k)|/d = \rho_k(y_i; z)s_i$ for $i = 1, \dots, n$, differs from s in that the entries of the nonpoor are set to zero.

The first poverty measure defined in Alkire and Foster (2011a) is the adjusted headcount ratio $M_0 = M_0(y; z) = \mu(g^0(k))$, or the mean of the censored deprivation matrix. It is the total value of all deprivations experienced by the poor as a share of the maximum total value of deprivations that would be obtained if everyone were fully deprived. Equivalently, $M_0 = \mu(s(k))$, or the mean of the censored vector of deprivation scores. M_0 can be expressed as the product of two intuitive partial indices, the headcount ratio H and the average intensity of poverty, denoted A . The headcount ratio arising from dual-cutoff identification is defined as $H = q/n$, where $q = \sum_{i=1}^n \rho_k(y_i; z)$ is total number of poor persons identified by ρ_k . The average intensity is given by $A = |s(k)|/q = (|g_1^0(k)| + \dots + |g_n^0(k)|)/(qd)$. It is easy to show that

$$M_0 = (|g_1^0(k)| + \dots + |g_n^0(k)|)/(nd) = HA \quad (1)$$

which offers a decomposition of the measure by population. This expression sums up terms in $g^0(k)$ horizontally across dimensions and then vertically across persons. In addition, M_0 can be broken down by dimension as

$$M_0 = (|g_{\cdot 1}^0(k)| + \dots + |g_{\cdot d}^0(k)|)/(nd) \quad (2)$$

which instead sums up terms in $g^0(k)$ vertically across persons and then horizontally across dimensions. We will return to these two expressions below.

Alkire and Foster (2011a) also define other measures that require more from the data – namely that each variable is cardinal – which ensures that the normalized gaps $g_{ij} = (z_j - y_{ij})/z_j$ of the poor are

¹³ Note that in the case of union identification, the censored and original deprivation matrices are identical.

meaningful. In this case, the censored deprivation matrix $g^0(k)$ can be replaced by the matrix $g^\alpha(k)$, having as its typical entry $g_{ij}^\alpha(k) = g_{ij}^0(k)((z_j - y_{ij})/z_j)^\alpha$ for a given $\alpha > 0$. A class of multidimensional poverty measures can then be defined by $M_\alpha(y; z) = \mu(g^\alpha(k))$ for $\alpha \geq 0$. In particular, the adjusted poverty gap M_1 is sensitive to the depth of deprivation in each dimension, while the adjusted FGT (or squared gap) M_2 emphasizes the largest normalized gaps and is sensitive to the Kolm type of multidimensional inequality in the distribution of achievements. Since our present concern is with measures that satisfy Ordinality, we focus on M_0 in what follows.

3. Properties

The properties of a poverty measure specify the patterns in the underlying data the measure should ignore, the aspects it should highlight, and the kinds of policy questions it can be used to answer. This section presents properties for multidimensional poverty measures, focusing first on the traditional properties satisfied by M_0 or, more precisely, by the methodology $\mathcal{M}_{k_0} = (\rho_{k_0}, M_0)$ since properties are, in fact, joint restrictions on identification and aggregation. Only general descriptions of these properties are provided here; precise definitions and verifications can be found in Alkire and Foster (2011a). Two additional properties of \mathcal{M}_{k_0} that were previously discussed, but have not yet received a rigorous treatment, are defined: *Ordinality*, which ensures that the measure can be meaningfully applied to ordinal data, and *Dimensional Breakdown*, which allows poverty to be broken down by dimension after identification. We conclude with a new property – *Dimensional Transfer* – that ensures that poverty is sensitive to a form of Atkinson-Bourguignon inequality among the poor.

The properties of multidimensional poverty methodologies can be divided into the categories of *invariance*, *subgroup*, and *dominance* properties. Invariance properties isolate aspects of the data that should *not* be measured. They include *Symmetry* (invariance to permutations of achievement vectors across people), *Replication Invariance* (invariance to replications of achievement vectors across people), *Deprivation Focus* (invariance to an increment in a nondeprived achievement), and *Poverty Focus* (invariance to an increment in an achievement of a nonpoor person).

Next are the subgroup properties that connect poverty levels overall to levels obtained from data broken down by population subgroup or by dimension. Two of the key properties here are *Subgroup Consistency* (if poverty rises in a population subgroup and stays constant in the remaining population, while subgroup population sizes are unchanged, then overall poverty must rise) and *Subgroup*

Decomposability (overall poverty is a population-weighted sum of the poverty levels in population subgroups).

Finally, are the dominance properties that concern the aspects of the data that *should* be measured and ensure that the poverty level responds appropriately to certain changes in achievements. They include *Weak Monotonicity* (an increment in a single achievement cannot increase poverty) and *Weak Rearrangement* (a progressive transfer among the poor arising from an association-decreasing rearrangement cannot increase poverty).¹⁴

For the purposes of this paper, we will take the above set of four invariance properties, two subgroup properties, and two dominance properties as the eight *basic multidimensional properties*. Below we discuss another dominance property from Alkire and Foster (2011a), namely, *Dimensional Monotonicity*, which requires poverty to fall as a result of an increment that removes at least one deprivation from among the poor. We now present three additional properties of multidimensional measures – respectively, an invariance property, a subgroup property, and a dominance property – that are the special concern of this paper.

3.1 Ordinality

The basic data used to construct the achievement matrix are typically derived from circumstances and conditions that are easy to describe and understand but have no natural metric in which to be measured. The numbers assigned to the various achievement levels (and deprivation cutoffs) in this domain are in a real sense simply placeholders to convey information about underlying conditions and, in particular, whether they are conditions of deprivation.¹⁵ Note that this general line of argument may be true even for the cases where the variable has an “in-built” representation such as income or years of schooling, since the cardinalization that comes with the variable may not be the right one for reckoning gains and losses in the present context.¹⁶

¹⁴ See Alkire and Foster (2011a) for more precise definitions of these properties. A third dominance property of *Weak Transfer*, which requires the application of the same bistochastic matrix to each dimension, is not well suited for ordinal variables and will not be considered here.

¹⁵ In fact, categorical information is all that is necessary in the present context. Even if the deprived achievements cannot be ranked one against the other, and the same is true for the achievements in the non-deprived category, one could use any numerical assignment that would correctly separate achievements into the deprived or non-deprived categories, with the deprivation cutoff being set at an appropriate value in between. The functions used below in the definition of equivalent representation need only preserve the categorical allocations.

¹⁶ For a fuller treatment of scales and measurement see Stevens (1946), Sen (1973, 1997), Alkire *et al.* (2015), and the references therein.

We say that $(y'; z')$ is obtained from $(y; z)$ as an *equivalent representation* if there exist increasing functions $f_j: R_+ \rightarrow R_+$ for $j = 1, \dots, d$ such that $y'_{ij} = f_j(y_{ij})$ and $z'_j = f_j(z_j)$ for all $i = 1, \dots, n$ and every $j = 1, \dots, d$. In other words, an equivalent representation assigns a different set of numbers to the same underlying basic data while preserving the original order. The methodology \mathcal{M}_{k_0} satisfies the following invariance property, which embodies the concern that the measure should be independent of the way the underlying data are represented.¹⁷

Ordinality: Suppose that $(y'; z')$ is obtained from $(y; z)$ as an equivalent representation. Then the methodology $\mathcal{M} = (\rho, M)$ satisfies $\rho(y'_i; z') = \rho(y_i; z)$, for all i , and $M(y'; z') = M(y; z)$.

To see that \mathcal{M}_{k_0} satisfies this property, note that the dimensions in which person i is deprived are unchanged between $(y'; z')$ and $(y; z)$, since the monotonic transformation ensures that $y'_{ij} < z'_j$ whenever $y_{ij} < z_j$. Consequently, the deprivation count is unchanged, which ensures that $\rho_k(y'_i; z') = \rho_k(y_i; z)$ for all i . It follows that the associated censored deprivation matrices are identical, so that their means are the same, and hence $M_0(y'; z') = M_0(y; z)$. Note that since the gap and squared gap matrices are typically very different for equivalent representations, the methodology $\mathcal{M}_{k\alpha} = (\rho_k, M_\alpha)$ violates Ordinality for $\alpha = 1$, $\alpha = 2$, and indeed any $\alpha > 0$. Each of these makes use of cardinal information on the depth of deprivations.

3.2 Dimensional Breakdown

Multidimensional poverty by definition has multiple origins, and it is useful for policy purposes to have a method of gauging how each dimension contributes to overall poverty. For example, information on contributions of dimensional deprivations could help in the allocation of resources across sectors and the design of specific or multisectoral policies to address poverty; monitoring progress dimension by dimension can help clarify the underlying sources of progress.¹⁸ A thoroughgoing decomposition of poverty by dimension would require overall poverty to be a weighted average of dimensional components, each of which is a function of that dimension's distribution of

¹⁷ We might imagine a weaker ordinality requirement that would only require the ordering, and not necessarily the measured level of poverty, to be preserved by equivalent representations.

¹⁸ Each of these examples has been used in practice: for example, in Colombia (CONPES 2012, Angulo *et al.* 2013) and in Costa Rica (Govt. of Costa Rica INEC 2015). Naturally the translation from measure to policy response requires additional analysis, as deprivations are often interconnected.

achievements only, without reference to achievements in the other dimensions. For example, Chakravarty, Mukherjee, and Ranade (1998) propose the following property:¹⁹

Factor Decomposability: There exist $v_j > 0$ summing to one, and component functions $m_j: Y_j \times R_{++} \rightarrow R_+$ such that

$$M(y; z) = v_1 m_{.1}(y_{.1}; z_1) + \dots + v_d m_{.d}(y_{.d}; z_d) \quad \text{for } y \text{ in } Y. \quad (3)$$

This property was originally constructed for methodologies using a union identification approach, so that by definition every deprivation is a poor person's deprivation. With a different identification, such as the general dual-cutoff approach, being deprived in a dimension does not automatically mean that a person is poor. Instead, this depends on the person's achievements in *other* dimensions through the identification function. And since a deprivation only contributes to poverty when the deprived person is poor, we must consider a form of breakdown by dimension that explicitly permits the component function to depend on information on who is poor. In our breakdown property, overall poverty is expressed as a weighted sum of dimensional components, but only after identification has taken place and the domain has been limited to the fixed set Y_r of achievement matrices with the same poverty status vector.²⁰

While Factor Decomposability placed no constraints on the component functions, one could argue that in order for a breakdown to be policy relevant, the component functions should reflect certain basic descriptive facts. For example, the contribution of a dimension to overall poverty would intuitively be zero if no poor persons were deprived in that dimension; while if one or more persons were both poor and deprived in the dimension, then the contribution would be positive. We say that $m_j(y_{.j}; z_j)$ is *normalized* if $m_j(y_{.j}; z_j) = 0$ when $y_{ij} \geq z_j$ for all poor persons i and $m_j(y_{.j}; z_j) > 0$ when $y_{ij} < z_j$ for some poor person i . This requirement will be incorporated into our breakdown property. As before, for any given dimension j , let Y_{rj} denote the set of all achievement distribution vectors y_j from some y in Y_r .

Dimensional Breakdown: For any given poverty status vector r , there exist $v_j > 0$ summing to one and normalized component functions $m_j: Y_{rj} \times R_{++} \rightarrow R_+$ for $j = 1, \dots, d$, such that

¹⁹ Chakravarty, Mukherjee, and Ranade (1998) also require the component functions to be identical, which seems a bit restrictive and is relaxed here. See also Chakravarty and Silber (2008) and Chakravarty (2009).

²⁰ This property allows the functional form of the breakdown to vary for every set of distributions having a different set of the poor – a less stringent and more general assumption than a full dimensional decomposition that requires the same functional form across all the subsets.

$$M(y; z) = v_1 m_1(y_{\cdot 1}; z_1) + \dots + v_d m_d(y_{\cdot d}; z_d) \quad \text{for } y \text{ in } Y_r. \quad (4)$$

In words, after identification has taken place and the poverty status of each person has been fixed, multidimensional poverty can be expressed as a weighted sum of dimensional components. The contribution of the deprivations in the j th dimension to overall poverty can then be viewed as $v_j m_j(y_j; z)/M(y; z)$.

In the case of the adjusted headcount ratio $\mathcal{M}_{k0} = (\rho_k, M_0)$, expression (2) can be stated equivalently as

$$M_0 = [(w_1/d) | g_{\cdot 1}^0(k) | / (nw_1) + \dots + (w_d/d) | g_{\cdot d}^0(k) | / (nw_d)]$$

which reduces to

$$M_0 = v_1 H_1 + \dots + v_d H_d \quad (5)$$

its version of expression (4), where the weights are $v_j = w_j/d$, or the value of deprivation j over the sum of the deprivation values, and the component functions are the censored headcount ratios H_j , or the percentage of the population that is both deprived in dimension j and poor. In other words, M_0 is a weighted average of the censored headcount ratios. Notice that H_j depends on the distribution of the other dimensional achievements, since all are needed to determine whether a person is poor. However, the entries in column $g_{\cdot j}^0(k)$ can be expressed as $g_{ij}^0(k) = g_{ij}^0 \rho_k(y_i; z)$ and hence depend on the other dimensional achievement levels *only through the identification function*. This ensures that when we restrict consideration to distributions having the same fixed poverty status vector r , the term reduces to $g_{ij}^0(k) = g_{ij}^0 r_i$ and hence depends only on the achievements in dimension j , as required by Dimensional Breakdown.

It can likewise be shown that every adjusted FGT methodology $\mathcal{M}_{k\alpha} = (\rho_k, M_\alpha)$ for $\alpha > 0$ presented in Alkire and Foster (2011a) satisfies Dimensional Breakdown. On the other hand, the multidimensional headcount ratio for any identification apart from the intersection approach must violate the property. For in this case there would be a dimension j in which a poor person need not be deprived. If everyone went from being deprived in all dimensions to being deprived in all but j , the normalized component function $m_j(y_j; z)$ would have to fall, but H would remain unchanged, violating Dimensional Breakdown. In the special case of intersection identification, intensity A is 1 and so the multidimensional headcount ratio H becomes M_0 , which satisfies Dimensional Breakdown.

3.3 Dimensional Transfer.

Transfer properties are motivated by the idea that poverty should be sensitive to the level of inequality among the poor, with greater inequality being associated with a higher (or at least no lower) level of

poverty.²¹ But which notion of inequality should be used in the multidimensional context? As noted in the introduction, there are two concepts in common use, one due to Kolm (1977) and another from Atkinson and Bourguignon (1982). The first is based on a definition of a progressive transfer as a “common smoothing,” whereby each dimensional distribution is transformed using the same bistochastic matrix. However, for this form of inequality to be meaningful, each dimensional variable would need to exhibit properties that are at odds with Ordinality.²²

The second inequality concept is based on a specialized transfer called a *rearrangement*, in which two persons switch achievements in certain dimensions. The role of a progressive transfer in this context is played by an *association-decreasing rearrangement*, in which the achievement vectors of the two persons are initially ranked by vector dominance (so that one person has no less in each dimension than the other person and more in one) and then after the rearrangement their achievement vectors cannot be ranked (so that one person has more in one dimension and the other has more in a second dimension). In symbols, we say that y' is obtained from y by an association-decreasing rearrangement if (a) both y' and y have the same population size; (b) there exist persons h and i such that for each $j = 1, \dots, d$ we have $\{y'_{hj}, y'_{ij}\} = \{y_{hj}, y_{ij}\}$, while the achievements for all other persons are unchanged; and (c) $y_i > y_h$, while neither $y'_i > y'_h$ nor $y'_h > y'_i$. This transformation can be interpreted as a progressive transfer in that it transforms an initial “spread” between two persons – a spread represented by the dominance between achievement vectors – into a moderated situation where neither person has unambiguously more than the other. The overall achievement levels in society are unchanged, but the correlation between them (and hence inequality) has been reduced.

Since this form of transfer involves a permutation and not an algebraic averaging of two persons’ dimensional achievements, it can be applied to ordinal data and is in principle consistent with the Ordinality property. The resulting transfer axiom for multidimensional poverty measures typically specifies that the persons involved in the rearrangement are poor. For example, the *weak rearrangement* axiom, as defined in Alkire and Foster (2011a), requires poverty not to rise as a result of an association-decreasing rearrangement among the poor. Note that this axiom – like all related axioms in the literature – is weak in that it does *not* ensure that poverty must strictly fall. It rejects the most problematic measures for which poverty can be “alleviated” by increasing inequality among the

²¹ See Sen (1976), Foster and Sen (1997), and Alkire *et al.* (2015).

²² The transformed achievement levels in a dimension are weighted averages of initial levels and hence depend on the cardinal representation of variables, which goes against Ordinality. In particular, after such a transformation, a person might be seen as poor under one cardinalization and nonpoor under a second. Measures applicable to ordinal variables cannot depend on the inequality level arising from a given cardinalization and, like M_0 , are independent of this form of inequality among the poor.

poor but at the same time allows measures to be entirely insensitive to progressive rearrangements among the poor.²³

A natural question to ask is whether a new version of this transfer axiom can be formulated that would, in certain circumstances, require poverty to strictly fall in response to a decline in inequality among the poor. One minimalist approach is to restrict consideration to cases where the association-decreasing rearrangement among the poor involves achievement levels that are on either side of deprivation cutoffs – thus affecting the distribution of deprivations as well. A *dimensional rearrangement among the poor* is an association-decreasing rearrangement among the poor (in achievements) that is simultaneously an association-decreasing rearrangement in *deprivations*. Recalling the definition of an association-decreasing rearrangement, we say that y' is obtained from y by a dimensional rearrangement among the poor if it satisfies (a)–(c) plus (d) $g_h^0 > g_i^0$ while neither $g_i^{0'} > g_h^{0'}$ nor $g_h^{0'} > g_i^{0'}$. In other words, the initial deprivation vectors (and achievement vectors) are ranked by vector dominance, while the final deprivation vectors (and achievement vectors) are not.²⁴ The extra condition ensures that the person with lower achievement levels is actually deprived in two or more dimensions for which the other person is not and that, through the rearrangement, one or more of these deprivations (but not all) are traded for non-deprived levels. The following transfer property for multidimensional poverty measures requires poverty to decrease when there is a dimensional rearrangement among the poor.

Dimensional Transfer: If y' is obtained from y by a dimensional rearrangement among the poor, then $M(y'; z) < M(y; z)$.

This axiom does not apply to cases where the association-decreasing rearrangement leaves deprivations unaffected; instead it requires the two persons to switch deprivations as well as achievements.

Note that this axiom is analogous to the axiom of Dimensional Monotonicity found in Alkire and Foster (2011a), which provides conditions under which a decrement in a dimensional achievement of a poor person must strictly raise the poverty level, namely, whenever the deprivation cutoff is crossed

²³ Such is the case of the headcount ratio H (since the number of poor persons is unchanged) and the adjusted headcount ratio M_0 (since the number of deprivations among the poor is also unchanged). Note that in order for an association-decreasing rearrangement among the poor to exist, there must be at least three dimensions; otherwise, it would be impossible to have vector dominance initially and the absence of vector dominance subsequently.

²⁴ The vector dominance in deprivations is converse to the vector dominance in achievements so that the person with lower achievements has more deprivations. In order to construct such a rearrangement, it must be possible to remove (or add) a deprivation from some poor person without altering his or her poverty status; this rules out intersection identification, for example, since a person must be deprived in all dimensions to be poor.

and the person becomes deprived in that dimension. Poverty must strictly rise as a result of such a *dimensional decrement among the poor* – which alters the achievement vector of a poor person so that there is vector dominance (upwards) in deprivations as well as vector dominance (downwards) in achievements.²⁵

A dimensional rearrangement among the poor does not affect the number of poor persons, and neither does a dimensional decrement among the poor. Consequently the headcount ratio $\mathcal{H}_k = (\rho_k, H)$ violates both Dimensional Transfer and Dimensional Monotonicity. In contrast, a dimensional increment among the poor decreases the average intensity of poverty A and hence $M_0 = HA$, which ensures that the adjusted headcount ratio $\mathcal{M}_{k0} = (\rho_k, M_0)$ satisfies Dimensional Monotonicity. But since a dimensional rearrangement among the poor leaves both H and A unchanged, \mathcal{M}_{k0} just fails to satisfy Dimensional Transfer.²⁶ The adjusted headcount ratio is always insensitive to this form of inequality among the poor. The next section explores the possibility of constructing measures that satisfy Dimensional Transfer.

4. New Measures from Old

The task at hand is to construct new methodologies that follow \mathcal{M}_{k0} in being able to be applied to ordinal data but, unlike \mathcal{M}_{k0} , are sensitive to inequality among the poor. In this section we maintain a dual-cutoff approach to identification, which is reasonably flexible and consistent with Ordinality, and search for a methodology satisfying Ordinality and Dimensional Transfer. We begin by altering the adjusted headcount measure M_0 to obtain such a measure and then expand the range of possibilities considerably using a novel way of constructing multidimensional poverty measures from unidimensional poverty measures – a process that is of interest in its own right.

Alkire and Foster (2011a) applied a simple power transformation to the individual poverty function from their cardinal measures, M_α for $\alpha > 0$, to obtain altered measures that would be sensitive to inequality across dimensions.²⁷ When the same transformation is applied to M_0 , it yields a methodology that satisfies Ordinality and Dimensional Transfer. For any power $\gamma \geq 0$, let $s^\gamma(k)$ be the vector

²⁵ Note that in order for a dimensional decrement among the poor to exist, there must be at least two dimensions as well as a way for a deprivation to be added to a poor person. With one dimension, a decrement in a poor person's level of achievement cannot alter the person's deprivations; likewise, intersection identification rules out adding a deprivation to a poor person – every poor person is already deprived in all dimensions.

²⁶ Because A is the average share of deprivations poor people experience, a rearrangement of the same set of deprivations among the same set of poor persons does not change A .

²⁷ See Alkire and Foster (2011a, p. 485).

obtained from $s(k)$ by raising each positive entry to the power γ , so that $s_i^\gamma(k) = (s_i)^\gamma$ when person i is poor, and $s_i^\gamma(k) = 0$ when i is not. For example, $s^0(k)$ is the identification vector r (which has value 1 if i is poor and 0 if i is not), $s^1(k)$ is the censored vector of deprivation scores $s(k)$, while $s^2(k)$ is the censored vector of squared deprivation scores. We define the M -gamma measure by $M_0^\gamma = \mu(s^\gamma(k))$ for $\gamma \geq 0$. Note that $M_0^0 = H$ is the multidimensional headcount ratio, $M_0^1 = M_0$ is the usual adjusted headcount ratio, while the measure M_0^2 obtained from $s^2(k)$ will be called the *squared count* measure.²⁸ The M -gamma measures M_0^γ for $\gamma > 1$ place greater emphasis on persons with higher deprivation scores, which leads to the following result.

Proposition 1. The methodology $\mathcal{M}_0^\gamma = (\rho_k, M_0^\gamma)$ satisfies Ordinality and Dimensional Transfer for $\gamma > 1$.

Proof. To verify Ordinality, suppose that $(y'; z')$ is obtained from $(y; z)$ as an equivalent representation. Then $(y'; z')$ and $(y; z)$ have the same deprivation matrix and hence the same deprivation counts for all i , from which it follows that $\rho_k(y'_i; z'_i) = \rho_k(y_i; z_i)$. This implies that each person's censored deprivation score is also the same, which means that $M_0^\gamma(y'; z') = M_0^\gamma(y; z)$ for every $\gamma > 1$. Thus, \mathcal{M}_0^γ satisfies Ordinality.

Now turning to Dimensional Transfer, suppose that y' is obtained from y by a dimensional rearrangement among the poor, so that (a) - (d) hold for persons h and i . We want to show that $M_0^\gamma(y; z) > M_0^\gamma(y'; z)$ or, equivalently, that $|s^\gamma(k)| > |s^\gamma(k)'|$. By definition $s(k)'$ and $s(k)$ differ only in coordinates h and i , and hence this condition reduces to $s_h^\gamma(k) + s_i^\gamma(k) > s_h^\gamma(k)' + s_i^\gamma(k)'$ or

$$(s_h)^\gamma + (s_i)^\gamma > (s_h')^\gamma + (s_i')^\gamma \quad (6)$$

since h and i are poor. But notice that $s_h + s_i = s_h' + s_i'$ since a rearrangement does not alter the total deprivations in the population. Moreover, the vector dominance of initial deprivation vectors given in (d) ensures that $s_h > s_i$, while the subsequent absence of vector dominance ensures that both s_h' and s_i' lie strictly between s_h and s_i . In other words, the final pair of deprivation scores (s_h', s_i') can be obtained from the initial pair of scores (s_h, s_i) by a progressive transfer. Condition (6) follows immediately from standard convexity results and the fact that $\gamma > 1$. Thus, \mathcal{M}_0^γ satisfies Dimensional Transfer for $\gamma > 1$. ■

²⁸ In the case of union identification, M_0^γ corresponds to the measure of social exclusion proposed in Chakravarty and D'Ambrosio (2006) and used in Jayaraj and Subramanian (2009). As we note below, it also has obvious links to the FGT class of unidimensional measures.

The above proof shows how a dimensional rearrangement among the poor across two achievement matrices becomes a progressive transfer among the poor for the associated deprivation score vectors, which has the effect of lowering poverty when it is measured as the average of the deprivation scores to the power $\gamma > 1$. Note that a person's deprivation score resembles the normalized poverty gap in the unidimensional world, so that the M -gamma measure M_0^γ has a form analogous to an FGT index, where γ is the power on the normalized gaps. We can build upon this insight to show how other multidimensional poverty measures satisfying Ordinality and Dimensional Transfer might be derived from unidimensional poverty measures. But before doing this, we pause to review the basic structure of unidimensional methods.

Unidimensional methods

Unidimensional poverty measurement has both an identification step and an aggregation step, with a simple method of identifying the poor based on a poverty cutoff separating the poor and nonpoor populations. An *identification function* $\varphi: R_+ \rightarrow \{0,1\}$ will be used here to indicate when a person is poor: a value of $\varphi(x_i) = 1$ signals that person i is poor, while a value of $\varphi(x_i) = 0$ means that i is nonpoor. As stressed by Donaldson and Weymark (1986), identification methods can differ both in the poverty cutoff $\pi \geq 0$ and in whether a person at π is considered to be nonpoor or poor.²⁹ One approach views the cutoff as the minimum acceptable level and regards all persons having incomes *below* π as poor. It yields the identification function $\varphi_<$ defined by $\varphi_<(x_i) = 1$ for $x_i < \pi$, and $\varphi_<(x_i) = 0$ for $x_i \geq \pi$. A second definition, following Sen (1976), views the poverty line as the maximum unacceptable level and regards those with incomes *below or equal to* the cutoff as being poor. It yields the identification function φ_\leq defined by $\varphi_\leq(x_i) = 1$ for $x_i \leq \pi$, and $\varphi_\leq(x_i) = 0$ for $x_i > \pi$. The set of the poor associated with the identification function φ is denoted by $\Pi \subseteq \{1, \dots, n\}$.

Let X be the set of nonnegative income distributions of all population sizes. A unidimensional poverty measure $P: X \cdot R_+ \rightarrow R_+$ aggregates the data in $x = (x_1, x_2, \dots, x_n)$ from X into an overall level $P(x; \pi)$ of poverty given the identification function φ and its poverty line $\pi \geq 0$. Common examples include the FGT class, the Watts (1968) measure, and the Sen (1976) measures. With the help of this notation, the standard subgroup decomposable measures can be written as

$$P(x; \pi) = \frac{1}{n} \sum_{i=1}^n \varphi(x_i) p(x_i; \pi) \quad (7)$$

²⁹ This distinction between $\varphi_<$ and φ_\leq is relevant when the poverty measure has a discontinuity at the poverty line, such as exhibited by the headcount ratio.

where the *individual poverty function* $p: R_+ \cdot R_+ \rightarrow R$ gauges a person's poverty level but is effectively censored by φ at all nonpoor income levels.³⁰ For example, the headcount ratio can be written using the constant individual poverty function $p(x_i; \pi) = 1$, while the remaining FGT measures can be defined using $p(x_i; \pi) = ((\pi - x_i)/\pi)^\alpha$ for $\alpha > 0$. Notice that the choice of $\varphi_{<}$ or φ_{\leq} has an impact on the measured level of poverty whenever $p(\pi; \pi) > 0$ (as with the headcount ratio), while the choice has no impact whatsoever when $p(\pi; \pi) = 0$ (as with the remaining FGT indices). Combining both the identification and aggregation steps, we obtain the *unidimensional poverty methodology* $\mathcal{P} = (\varphi, P)$, where $\varphi(x_i)$ is the identification function and $P(x; \pi)$ is the poverty measure.

Properties for unidimensional poverty measures (or more precisely, methodologies) include *invariance* properties such as *Symmetry* (invariance to permutations of incomes), *Replication Invariance* (invariance to replications of incomes), and *Focus* (invariance to an increment in an income of a nonpoor person); *subgroup* properties like *Subgroup Consistency* (if poverty rises in a population subgroup and stays constant in the remaining population, while subgroup population sizes are unchanged, then overall poverty must rise) and *Subgroup Decomposability* (overall poverty is a population-weighted sum of the poverty levels in population subgroups); and the *dominance* properties of *Weak Monotonicity* (an increment in a single income cannot increase poverty) and *Weak Transfer* (a progressive transfer of income among the poor cannot increase poverty).³¹

For the purposes of this paper, we will take the above set of three invariance properties, two subgroup properties, and two dominance properties as the seven *basic unidimensional properties*. Two additional dominance properties can be defined that make use of the following forms of distributional changes. We say that x' is obtained from x by a *decrement among the poor* if there is a person i with $\varphi(x_i) = 1$ such that $x_i' < x_i$, while for all $h \neq i$ we have $x_h' = x_h$. In words, such a decrement involves a poor person losing income while all other incomes are unchanged. We say that x' is obtained from x by a *progressive transfer among the poor* if there are two persons h and i with $\varphi(x_h) = \varphi(x_i) = 1$ such that $x_h' - x_h = x_i - x_i' = \Delta > 0$ where $x_i - x_h > \Delta$, while for all other i' we have $x_i' < x_i$. In words, such a

³⁰ See expression (22) in Foster and Shorrocks (1991) or the continuous version (2) in Atkinson (1987). Since (7) separates out terms for aggregation and identification, it allows either form of identification to be used. Note that the value of p is irrelevant for nonpoor persons and that $P = 0$ when no one is poor.

³¹ For definitions, see Foster et al. (2013). Notice that the weak transfer property is “weak” by virtue of allowing a weak inequality; in Foster and Sen (1997) the weak transfer axiom refers to a limitation on which transfers are allowed and is equivalent to the Transfer property below. Another property commonly assumed is *Scale Invariance*, which implies that a doubling of all incomes and the poverty line will leave poverty unchanged. In our analysis, the poverty line will be fixed and hence the property is less relevant here.

transfer involves a richer poor person giving income to an even poorer person, but not so much that they switch incomes. The remaining properties are as follows.

Monotonicity: If x' is obtained from x by a decrement among the poor, then

$$P(x'; \pi) > P(x; \pi)$$

Transfer: If x' is obtained from x by a progressive transfer among the poor, then

$$P(x'; \pi) < P(x; \pi)$$

The properties specify that the poverty methodology must register an increase in poverty when there is a decrement among the poor and a decrease in poverty when there is a progressive transfer among the poor.

From unidimensional to multidimensional

Having reviewed the structure of unidimensional poverty measurement, we return to the multidimensional environment where a useful transformation from unidimensional poverty methodologies can be described. The process begins with an $n \times d$ achievement matrix y and its deprivation matrix g^0 . We define an *attainment matrix*, denoted by a^0 , that is complementary to the deprivation matrix in that it indicates when persons are *not* deprived. The attainment matrix a^0 has the typical element $a_{ij}^0 = w_j - g_{ij}^0$ for $i = 1, \dots, n$ and $j = 1, \dots, d$. In other words, $a_{ij}^0 = w_j$ whenever i is *not* deprived in j , and $a_{ij}^0 = 0$ whenever i is deprived in j . The *attainment count vector*, denoted by a , is the vector defined by $a_i = a_{i1}^0 + \dots + a_{id}^0$ for each $i = 1, \dots, n$. In other words, it gives an aggregate value of attainment for each person. Attainment counts can range between 0 and d , and when added to deprivation counts they always sum to d (so that $a_i + c_i = d$ for all i).

Now let $\mathcal{P} = (\varphi, P)$ be any unidimensional poverty methodology for which $0 \leq \pi \leq d$. Define the associated multidimensional poverty methodology $\mathcal{M}_p = (\rho_\varphi, M_P)$ by $\rho_\varphi(y; z) = \varphi(a_i)$ and $M_P(y; z) = P(a; \pi)$, where a is the attainment count vector associated with y given z and w . In other words, \mathcal{M}_p applies the unidimensional methodology \mathcal{P} to the attainment count distribution. In particular, \mathcal{M}_p identifies person i as being poor in y if $\varphi(a_i) = 1$, and it measures poverty in y as the unidimensional poverty level $P(a; \pi)$ in a , given the identification function and its poverty standard π . The resulting multidimensional poverty methodology \mathcal{M}_p will be called an *attainment count methodology* while the associated process of obtaining \mathcal{M}_p from \mathcal{P} will be called the *attainment count transformation*.

It is easy to see that the attainment count methodology uses a dual-cutoff identification from Alkire and Foster (2011a), where the poverty cutoff is given by $k = d - \pi$. For example if $\varphi = \varphi_{\leq}$ is being used in \mathcal{P} , then since $\rho_{\varphi}(y_i; z) = \varphi(a_i)$, it follows that person i is poor whenever $a_i \leq \pi$, hence $d - c_i \leq d - k$ or $c_i \geq k$ as required by the standard dual-cutoff identification function ρ_k . On the other hand, if \mathcal{P} uses $\varphi = \varphi_{<}$, then the identification function ρ_{φ} yields the *strict* dual-cutoff identification function ρ_k° where once again $k = d - \pi$, but a person is considered poor only if $c_i > k$.³² In either case, it is easy to show that the measure M_P is unaffected by an increase in the achievement of any person who is not deprived in that dimension (hence the methodology satisfies Deprivation Focus) and, since an equivalent representation preserves the attainment count matrix and distribution, both the set of poor and the measure of poverty will be unaffected (hence the methodology satisfies Ordinality). These two properties hold for \mathcal{M}_p no matter which methodology \mathcal{P} is used. Other multidimensional invariance properties are inherited from the associated unidimensional properties. For example it is easy to see that Symmetry and Replication Invariance immediately follow from their counterparts while Poverty Focus follows from the unidimensional Focus axiom. It is likewise immediate that the two multidimensional subgroup properties follow from their unidimensional versions. These results can be summarized as follows:

Proposition 2. The attainment count methodology \mathcal{M}_p employs a dual-cutoff identification function ρ_k (or ρ_k°) and satisfies Deprivation Focus and Ordinality. Moreover, if \mathcal{P} satisfies (i) Symmetry, (ii) Replication Invariance, (iii) Focus, (iv) Subgroup Consistency, or (v) Subgroup Decomposability, then \mathcal{M}_p satisfies the associated multidimensional property of (I) Symmetry, (II) Replication Invariance, (III) Poverty Focus, (IV) Subgroup Consistency, or (V) Subgroup Decomposability.

In words, any methodology produced by the attainment count transformation has a dual-cutoff identification function and is applicable to ordinal data. In addition, it transforms the basic invariance and subgroup properties of unidimensional measurement into the analogous properties of multidimensional measurement. Hence any standard unidimensional measure, such as the headcount ratio, the poverty gap ratio, or others, will produce a multidimensional poverty measure with these desirable properties. We shall see below that dominance properties, too, are reproduced through the

³² Both versions of the dual-cutoff approach are considered by Alkire and Foster (2011a) although they focus on the version based on the inequality $c_i \geq k$. Note that the value $\pi = d$ in the presence of φ_{\leq} yields the trivial case where all person are always identified as being poor, and hence we typically restrict consideration to $0 \leq \pi < d$ in this case. Likewise the value $\pi = 0$ in the presence of $\varphi_{<}$ yields the trivial case where no one is ever identified as being poor, and hence we typically assume $0 < \pi \leq d$ in this case.

transformation. But first we consider several examples of attainment count methodologies to see how the transformation works in practice.

Example 1: Headcount Ratio Consider $\mathcal{P} = (\varphi, P)$ where $P = P_0$ is the unidimensional headcount ratio given φ . Then for any z and w , the resulting attainment count methodology $\mathcal{M}_p = (\rho_\varphi, M_P)$ is made up of a dual-cutoff identification function and a multidimensional headcount measure. In particular, if $\varphi = \varphi_\leq$ then ρ_φ is ρ_k , or the standard dual-cutoff identification function with poverty cutoff $k = d - \pi$, while M_P is $H(y; z) = (1/n) \sum_i \rho_k(y; z)$, or the usual multidimensional headcount ratio of Alkire and Foster (2011a). On the other hand, if $\varphi = \varphi_<$ then ρ_φ is ρ_k° , the strict dual-cutoff identification function, while M_P is $H(y; z) = (1/n) \sum_i \rho_k^\circ(y; z)$, or an alternative multidimensional headcount ratio used, for example, by Mexico.³³ When π is close enough to 0 (and hence k is close to d) the methodology has intersection identification, where poor persons are deprived in all dimensions at once and hence have no attainments at all. Likewise, when π is close enough to d (and hence k is close to 0) then it has union identification, in which all poor persons have at least a single deprivation.³⁴

Example 2: Adjusted Headcount Ratio Suppose that P takes the form of the decomposable index $\lambda P_0 + (1-\lambda)P_1$ as given in Foster and Shorrocks (1991), where P_0 is the unidimensional headcount ratio, P_1 is the poverty gap ratio, and $\lambda = k/d$. It is an easy matter to show that P is the *augmented poverty gap ratio* defined as $P_1^d(x; \pi) = \frac{1}{n} \sum_{i=1}^n \varphi(x_i) p^d(x_i; \pi)$, which uses the poverty value function $p^d(x_i; \pi) = (d - x_i)/d$ instead of $(\pi - x_i)/\pi$ used by the standard poverty gap ratio.³⁵ Given the identification function $\varphi = \varphi_\leq$ and applying P_1^d to the attainment distribution a , we obtain the multidimensional methodology $\mathcal{M}_{k_0} = (\rho_k, M_0)$ or the adjusted headcount ratio methodology of Alkire and Foster (2011a). To see this, note that when the poor are identified using $\varphi_\leq(a_i; \pi)$ and measured using $P_1^d(a; \pi)$, the overall poverty level is found by averaging the terms $\varphi_\leq(a_i; \pi)(d - a_i)/d$ across all $i = 1, \dots, n$. But for every poor person i (with $a_i \leq \pi$, and hence $c_i \geq k$) this term reduces to $c_i/d = c_i(k)/d$, while for all nonpoor i (with $a_i > \pi$, and hence $c_i < k$) it is $0 = c_i(k)/d$. The average of $c_i(k)/d$ across all i is the same as the average of the

³³ The Mexican technology has seven dimensions with effective deprivation values of $w_1 = 7/2$ on income deprivation, $w_j = 7/12$ for $j = 2, \dots, 7$ or the social deprivations, and a strict poverty cutoff of $k = 7/2$. See CONEVAL (2009).

³⁴ More precisely, intersection identification is obtained when π is below w_{min} , so that k is above $d - w_{min}$, where w_{min} is the smallest deprivation value, while union identification is obtained when π is above $d - w_{min}$, so that k is below w_{min} .

³⁵ This form of poverty measure is useful for measuring ultra-poverty, where a poverty cutoff is used to identify the ultra-poor, while a higher poverty standard is used in the aggregation function. See Foster and Smith (2016).

entries in the censored deprivation matrix $g^0(k)$, and hence the poverty measure is just the adjusted headcount ratio $M_0 = \mu(g^0(k))$.³⁶

Example 3: An Alternative Adjusted Headcount Ratio Suppose that $P = P_1$ is the traditional poverty gap ratio, which evaluates the net shortfalls from the poverty cutoff. Applying the identification function $\varphi = \varphi_<$ and measure P_1 to the attainment distribution a , we obtain the multidimensional methodology $\mathcal{M}_p = (\rho_k^\circ, M'_0)$ in which person i is identified as poor if $c_i > k$ and the poverty measure is $M'_0(y; z) = (1/n)\sum_i (c_i/d')\rho_k^\circ(y_i; z)$, where $c_i' = c_i - k$ is the net deprivation count above the poverty cutoff for person i and $d' = d - k$ is the maximum possible net deprivation count above the poverty cutoff. In words, $M'_0(y; z)$ is the alternative adjusted headcount ratio that measures a poor person's intensity of deprivation using the net deprivation share c_i'/d' . To see this, note that $P_1(a; \pi) = (1/n)\sum_i \varphi_<(a_i)(\pi - a_i)/\pi$ where $(\pi - a_i)/\pi = (d-k-a_i)/(d-k) = (c_i-k)/(d-k) = c_i'/d'$ and $\varphi_<(a_i) = \rho_k^\circ(y_i; z)$, so that clearly $P_1(a; \pi) = M'_0(y; z)$. The associated measure of intensity of poverty $A' = M'_0/H$ can be expressed as the income gap ratio $I = P_1/P_0$ applied to the attainment distribution a . In the Mexican example, for instance, the deprivation value for income is $w_1 = 7/2$ and the poverty cutoff is $k = 7/2$, so the identification condition $c_i > k$ ensures that all poor persons are deprived in income and at least one additional non-income dimension. The intensity of a poor person's deprivation is measured by the share of all possible *non-income* deprivations a person has, while a nonpoor person has zero intensity. Averaging across all persons yields the alternative adjusted headcount ratio M'_0 .

The above examples show that several key multidimensional methodologies used in practice are in fact attainment count methodologies, including the headcount and adjusted headcount methodology from Alkire and Foster (2011a) (which alone of the three satisfies the Dimensional Breakdown property), employed in the global MPI and other country measures, and the headcount and adjusted headcount methodologies that underlie Mexico's official measure. Proposition 2 further shows that each satisfies Ordinality and the other basic invariance and subgroup properties for multidimensional poverty measures. However, it can also be shown that each measure is unaffected by a dimensional rearrangement among the poor and, hence, just violates Dimensional Transfer. We must go beyond headcount ratios and adjusted headcount ratios to satisfy this dominance property. The next result

³⁶ This is the form used in the UNDP's Multidimensional Poverty Index (MPI) and in Colombia's official multidimensional poverty measure. See also Alkire and Santos (2010, 2014). In addition, the associated measure of intensity (or breadth) of poverty $A = M_0/H$ can be derived by applying the augmented income gap ratio $I^d = P_1^d/P_0$ to the distribution of attainments a .

shows how the attainment count transformation converts dominance properties for unidimensional measures into dominance properties for multidimensional measures, including Dimensional Transfer.

Proposition 3. Let \mathcal{M}_p be the attainment count methodology generated by a unidimensional methodology \mathcal{P} . If \mathcal{P} satisfies the unidimensional property of (a) Weak Monotonicity, (b) Weak Transfer, (c) Monotonicity, or (d) Transfer, then \mathcal{M}_p satisfies the associated multidimensional property of (A) Weak Monotonicity, (B) Weak Rearrangement, (C) Dimensional Monotonicity, or (D) Dimensional Transfer.

Proof. See Appendix.

The main ideas behind this result are as follows. Weak Monotonicity (A) and Dimensional Monotonicity (C) consider the effects of changing a single achievement level for a person i . If the person begins and stays deprived (or nondeprived) in that dimension, then a_i is unchanged and so is measured poverty. If an increment crosses the deprivation cutoff, then a_i rises and Weak Monotonicity (a) ensures that poverty does not rise. If a decrement crosses the deprivation cutoff and the person is already poor, then a_i falls and Monotonicity (c) ensures that poverty increases and hence that Dimensional Monotonicity (C) is satisfied. In contrast, Weak Transfer (B) and Dimensional Transfer (D) consider the impact of an association-decreasing rearrangement among two poor persons, say i and i' . This is a situation where one of the two (say i) has an achievement vector that is initially larger than that of the second (say i'), but after the rearrangement neither has more of every achievement. If both persons are deprived (or nondeprived) in the dimensions that have been rearranged, then a_i and $a_{i'}$ will be unchanged and hence poverty is unaffected. If the persons switch deprivations, as with a dimensional rearrangement among the poor, then a_i falls by the same amount that $a_{i'}$ rises, and hence the new attainment vector is obtained from the old by a progressive transfer. Weak Transfer (b) ensures that poverty does not rise with a progressive transfer, and hence we obtain Weak Transfer (B). Likewise, Transfer (d) ensures that this progressive transfer among the poor decreases poverty, and hence we obtain Dimensional Transfer (D). It is this link between association-decreasing rearrangements in *achievements* and progressive transfers in *attainments* that underlies these last results.

A main lesson to be drawn from this section is that since many existing unidimensional measures satisfy Transfer, it is easy to construct multidimensional measures that satisfy Dimensional Transfer and are thus sensitive to inequality among the poor. For example, the Watts measure W which takes the form of expression (7) with $p_0(x_i; \pi) = \ln(\pi/x_i)$, the Clark-Hemming-Ulph class C_β given by $p_\beta(x_i; \pi) = 1 - (x_i/\pi)^\beta$ for $\beta < 1$ and $\beta \neq 0$, or the FGT class P_α with $p_\alpha(x_i; \pi) = ((\pi - x_i)/\pi)^\alpha$ for the

range $\alpha > 1$, all satisfy Transfer and thus generate multidimensional measures satisfying Dimensional Transfer (and other properties including Ordinality and Subgroup Decomposability).³⁷ Each of these measures builds upon the alternative “net shortfall” version M'_0 of the adjusted headcount ratio. The standard adjusted headcount ratio M_0 , which measures intensity using the full range of deprivations a poor person experiences, is the basis for a second group of measures applying *augmented* versions of the unidimensional measures, which replace π with d in the individual poverty functions. For example, consider the augmented FGT indices P_α^d generated by $p_\alpha^d(x_i; \pi) = ((d - x_i)/d)^\alpha$ for $\alpha > 0$ and $p_\alpha^d(x_i; \pi) = 1$ for $\alpha = 0$. Since $P_0^d = P_0$, the multidimensional measure obtained using P_0^d is the same multidimensional headcount ratio H as before, while the standard adjusted headcount ratio M_0 is obtained using the augmented poverty gap ratio P_1^d . For $\alpha > 1$ the augmented FGT measure P_α^d satisfies Transfer, Monotonicity, and the seven basic unidimensional properties; consequently the multidimensional poverty measure it generates satisfies Dimensional Transfer, Dimensional Monotonicity, and the eight basic multidimensional properties. Indeed, the attainment count transformation of the unidimensional methodology $(\varphi_\leq, P_\alpha^d)$ for $\alpha \geq 0$ generates the M -gamma methodology $\mathcal{M}_0^\gamma = (\rho_k, M_0^\gamma)$ for $\gamma \geq 0$ where $\gamma = \alpha$. The M -gamma class of multidimensional measures inherits its properties, including its sensitivity to inequality, from the class of FGT measures used to generate it. It is a natural generalization of the FGT measures to multidimensional poverty measures using ordinal data.

5. Possibilities

The attainment count transformation illustrates how easy it is to extend the traditional adjusted headcount ratio to obtain measures satisfying Dimensional Transfer and the basic properties of Ordinality and Subgroup Decomposability. But what has happened to the key property of Dimensional Breakdown, which allows dimensional contributions to overall poverty to be assessed in order to direct policy more effectively? Let us return to the class $\mathcal{M}_0^\gamma = (\rho_k, M_0^\gamma)$ for $\gamma \geq 0$, containing the headcount ratio H for $\gamma = 0$, the adjusted headcount ratio M_0 for $\gamma = 1$, and the squared breadth measures M_0^2 for $\gamma = 2$. While all of the M -gamma measures M_0^γ in the range $\gamma > 1$ satisfy Dimensional Transfer, it is easy to see that *none* of them satisfies Dimensional Breakdown. Take the measure M_0^2 for example.

³⁷ The Sen measure, both in its original discontinuous version from Sen (1976) and its updated continuous version from Foster and Sen (1997), yields a multidimensional poverty measure satisfying Dimensional Transfer and the remaining properties apart from Subgroup Decomposability and Subgroup Consistency.

The marginal value of an additional deprivation among the poor clearly depends on the breadth of deprivation experienced by the affected poor person. In contrast, Dimensional Breakdown requires the contribution of the dimension to overall poverty to be independent of the other dimensions (after identification). This conflict between Dimensional Transfer and Dimensional Breakdown is actually quite fundamental, as shown in the following result.

Proposition 4. Let methodology $\mathcal{M} = (\rho, M)$ satisfy Symmetry, and suppose that \mathcal{M} has a dimensional rearrangement among the poor. Then \mathcal{M} cannot simultaneously satisfy Dimensional Breakdown and Dimensional Transfer.

Proof. Suppose that \mathcal{M} satisfies Symmetry. Let y' be obtained from y by a dimensional rearrangement among the poor. By symmetry, without loss of generality we can assume that 1 and 2 are the poor persons involved in the rearrangement and that $y_1 > y_2$. Now let J be the set of all dimensions j that are unchanged in the rearrangement, so that both $y'_{1j} = y_{1j}$ and $y'_{2j} = y_{2j}$ hold. Let x be the achievement matrix obtained from y by lowering person 1's achievement level in each $j \in J$ to person 2's achievement level, leaving all remaining entries unchanged. Similarly construct x' from y' by lowering person 1's achievement level in each $j \in J$ to that of person 2. Person 1 remains poor in both x and x' , and so all four achievement matrices have the same poverty status vector r . By dimensional breakdown, then, there exist functions $m_j : Y_{rj} \cdot R_{++}^d \rightarrow R_+$ for $j = 1, \dots, d$, such that expression (3) holds for all matrices in Y_r . Applying (3) to y and y' yields

$$M(y'; z) - M(y; z) = \sum_{j \notin J} [m_j(y'_{.j}; z_j) - m_j(y_{.j}; z_j)]$$

since y' and y are the same in all dimensions of J . Applying (3) to x' and x yields

$$M(x'; z) - M(x; z) = \sum_{j \notin J} [m_j(x'_{.j}; z_j) - m_j(x_{.j}; z_j)]$$

for the analogous reason. By construction, it follows that $y'_{.j} = x'_{.j}$ and $y_{.j} = x_{.j}$ for all $j \in J$ and hence that $M(y'; z) - M(y; z) = M(x'; z) - M(x; z)$. However, x' is simply a permutation of x (between persons 1 and 2) and so by Symmetry we have $M(x'; z) - M(x; z) = 0$. This implies that $M(y'; z) = M(y; z)$ and thus dimensional transfer is violated. ■

Proposition 4 shows how Dimensional Transfer directly conflicts with Dimensional Breakdown. The underlying idea is that the curvature required to obtain a lower level of multidimensional poverty from a rearrangement among the poor leads to a violation of the limited form of separability embodied in Dimensional Breakdown. Of course, if no such rearrangement were to exist – which could be the case for certain methodologies – Dimensional Transfer would technically hold but would be empty in

practice. We thus presume that there is at least one situation to which Dimensional Transfer applies and the contradiction follows. The conflict pertains to all multidimensional poverty methodologies, including those satisfying Ordinality as well as others that are meaningful only for cardinal variables.

Note that the separability requirement implicit in Dimensional Breakdown is less restrictive than the full factor decomposability requirement invoked by Chakravarty et al. (1998). Dimensional Breakdown restricts the measure on the domain associated with a fixed vector of the poor, whereas factor decomposability requires a dimensional decomposition to hold over the entire domain. Similarly, the separability assumption of Pattanaik, Reddy, and Xu (2012) is the ordinal version of factor decomposability and has no domain restrictions over the distributions being considered. Their main result, which they characterize as a “disturbing consequence,” shows how separability by dimension ensures that all rearrangements have no effect. While the context is different – multidimensional deprivation orderings rather than measures – and their assumption of separability is substantially broader in scope than Dimensional Breakdown, the intuition behind their result and Proposition 4 is the same. In contrast, Rippin (2013) makes the false claim that a multidimensional poverty measure can satisfy factor decomposability while remaining sensitive to dimensional rearrangements among the poor, and offers M_0^γ with union identification and $\gamma > 1$ as an example. Of course, as noted above, these measures violate factor decomposability as well as Dimensional Breakdown, and any claim to the contrary would contradict Pattanaik, Reddy, and Xu (2012).³⁸

Given the necessity of choosing between Dimensional Breakdown and Dimensional Transfer, how are empirical and operational studies of multidimensional poverty to proceed? There is by now a rich literature that shows how Dimensional Breakdown, as implemented through expression (2) for the adjusted headcount ratio, can enrich the informational content of multidimensional poverty. It can enhance monitoring by dimension across time and space and allow policies to be tailored to the composition of poverty. With the help of this property, M_0 becomes a tool for good governance through positive feedback loops that reward effective policies, and it becomes a tool for coordination among ministries who work together toward the common goal of reducing poverty, as noted in Angulo (2016). These and other benefits suggest that M_0 should remain in place as a key measure of multidimensional poverty.

³⁸ Rippin (2013) offers a confused explanation of how this impossible task is to be accomplished: The individual poverty function (which depends on all dimensions) is simply re-designated as an “identification function”. However, the aggregate measure and its properties are unaffected by this artifice. Factor decomposability does not hold. The result of Pattanaik, Reddy, and Xu (2012) applies. Unfortunately, the false claim that M_0^γ satisfies factor decomposition for $\gamma > 1$ has not yet been corrected. See for example Rippin (2015).

Dimensional Transfer is intended to ensure that a multidimensional poverty measure conveys information on the inequality of attainments among the poor. Our attainment count transformation provides several routes to depicting this information by drawing upon analogous approaches from unidimensional measures; the particular methodology used depends upon the purpose of the exercise and the pertinent questions for analysis. In traditional comparisons of international poverty, the three most common P -alpha measures P_0 , P_1 , and P_2 are often reported to convey information on income poverty's incidence, depth, and severity, respectively. The attainment count transformation suggests the analogous use of the three main M -gamma measures – namely the headcount ratio M_0^0 , the adjusted headcount ratio M_0^1 , and the squared count measure M_0^2 – to evaluate the incidence, breadth, and severity of multidimensional poverty, with M_0^1 containing information on intensity and M_0^2 containing information on inequality among the poor.

To directly monitor the inequality in the distribution of attainments among the poor, one could use relative measures like the squared coefficient of variation (underlying P_2) or the Gini coefficient (underlying the Sen measure) or absolute inequality measures like the variance. The size and distribution of attainments among the poor could also be evaluated using an “income standard” such as an Atkinson equally distributed equivalent income function or the Sen welfare index applied to the distribution of attainments among the poor. Given the theoretical link between unidimensional poverty, inequality, and income standards among the poor as emphasized in Foster et al. (2013), these multidimensional methods are closely related. They offer a window into a type of multidimensional inequality based on the breadth of attainments people experience, which becomes especially relevant when data are ordinal.

The distribution of attainments or deprivations among the poor can also be examined directly to reveal information on inequality. For example, we might plot the associated Lorenz curves or examine the various orders of stochastic dominance to see how a distribution is changing. One basic option is to partition the distribution of deprivations (or achievements) among the poor into *intensity bands* of observations and monitor the percentages in each. For example, using the global MPI, we might depict the percentages of the poor having deprivation scores in the range of 33–39% of deprivations, 40–49%, and so on to 100%, where the percentages of the poor across the ranges can be compared to see whether inequality is diminishing or advancing.³⁹ The comparisons can be further enriched by applying

³⁹ For empirical examples see Alkire and Seth (2013), who show transitions across population subgroups of subgroup members experiencing different intensity bands of poverty and Alkire Roche and Vaz (2015) who study results across 34 countries and their subnational regions for multiple poverty and deprivation cutoffs.

dimensional breakdown to the population in each intensity band and contrasting the dimensional composition of poverty experienced by those having different ranges of deprivation scores.

Consider the following example that uses the 2011 Demographic and Health Survey to compute the global MPI for Cameroon and its 12 subnational regions.⁴⁰ As compared to other developing countries, Cameroon has a medium level of multidimensional poverty with 46% of the population living in poverty and an MPI of 0.248. To emphasize the relationships among the three M -gamma measures, we multiply their decimal values by 100 to express poverty levels in hundredths so that the headcount ratio for Cameroon is 46 and the MPI is 24.8. The disparity in the incidence of poverty across regions is striking, with the headcount ratio ranging from 6.5 to 86.7 and the MPI ranging from 2.4 to 54.

Table 1 presents the dimensional breakdowns of the MPI for the regions and for the country as a whole. The censored headcount ratios reveal a great variety in the composition of poverty across the regions. Consider, for example, Adamaoua and Est, two regions close in population size and with similar levels of MPI (26.9 and 27.4, respectively). From the breakdown, we see that they have marked differences in the composition of poverty, with education deprivations of the poor in Adamaoua being far more widespread than in Est, health deprivations of the poor being somewhat more common, and living standard deprivations of the poor being a good deal less common. For example, the censored headcount ratio for school attendance is 20.5 in Adamaoua, which indicates that 20.5% of Adamaoua's population are MPI poor and live in households in which a child is not attending school up to the age at which they should complete class eight; the comparable figure in Est is 13. Likewise, poverty in Est shows higher living standard deprivations, with a censored headcount ratio for sanitation and water at 51 and 44, respectively, as compared with 25 and 24 for Adamaoua. The policies needed for responding to poverty in regions with similar MPI levels can vary in terms of allocation and sectoral emphases.

Table 2 lists poverty levels for Cameroon and its regions using the three main M -gamma measures, namely, the headcount ratio $M_0^0 = H$, the adjusted headcount ratio $M_0^1 = M_0$ discussed above, and the squared count measure M_0^2 . Column 1 gives the headcount ratio for Cameroon (namely $M_0^0 = 46$) along with its regional levels. The adjusted headcount ratio for Cameroon and its regions reappears in column 2, beginning with Cameroon's value of $M_0^1 = HA = 24.8$ for the country. It adjusts the headcount ratio by the breadth of poverty in the form of intensity $A = |s(k)|$. The squared count measure $M_0^2 = HA^2$ in column 3 instead adjusts the headcount ratio by the term $A^2 = |s^2(k)|$, which is sensitive to the

⁴⁰ The parameters of the global MPI used in this example are presented in Alkire and Robles (2015). Note that the MPI Table 1 includes the variance measures of inequality among the poor for every country, and all MPI country briefings include the analysis by intensity bands.

inequality of deprivations (and attainments) among the poor, yielding a level of $M_0^2 = 14.7$ for Cameroon. In Table 2, regions are ordered according to M_0^1 , and the rankings according to the three measures are quite similar. There is one reversal between Douala and Yaoundé when the headcount ratio M_0^0 is used instead of the adjusted headcount measure. This is due to the higher intensity of poverty in Yaoundé. There are two reversals (Sud-Ouest vs. Sud and Adamaoua vs. Est) when we move from the adjusted headcount ratio to the squared count measure M_0^2 . This is reflected in the higher inequality-sensitive intensity of poverty in Sud-Ouest and Adamaoua. Inequality-sensitive results like those found in Table 2 could be generated using any other unidimensional poverty measure satisfying the transfer property. However, Table 1 requires a measure like M_0 that satisfies dimensional breakdown.

Table 3 presents various measures of inequality among the poor using the truncated vector of deprivation scores s , simply for illustrative purposes. The absolute inequality measure of variance can also provide interesting comparisons across countries. See for example Seth and Alkire (2013). Descriptive methods of depicting inequality among the poor are readily available and can supplement a single inequality measure. For example, Table 4 divides the population of poor people into categories according to the deprivation score. Column 1 shows the percentage of poor persons who are deprived in 33.3% to 39.9% of the dimensions only – a low intensity band. Yet we can see that in Douala, 82% of poor people are in this category, and none have 50% or more deprivations. The example of Adamaoua and Est is, again, striking. A total of 48% of poor persons are deprived in 33.33 to 50% of the dimensions, but in Adamaoua, only 26% are in the lowest band and 22% are deprived in 40–49.9%, whereas in Est, 39% of poor people are in the lowest intensity band. At the high intensity end, Adamaoua again is visibly worse than Est. Such descriptive information might provide the intuition behind the observed shift in ranking using the inequality-adjusted measure M_0^2 . While information on inequality can help identify high inequality situations where most deprivation scores are at the extremes, it should be remembered that reducing *inequality* among the poor is not the main objective: reducing *poverty* is.

Table 1: Dimensional Breakdown for Cameroon and Its Regions: MPI and Censored Headcount Ratios

	MPI (M_0)	Education		Health		Living standards					
		Schooling	School Attendance	Child Mortality	Nutrition	Electricity	Improved Sanitation	Safe Water	Floor	Cooking Fuel	Assets
Cameroon	24.8	16.7	18.4	27.4	18.3	37.3	34.7	28.9	34.5	45.5	23.0
Douala	2.4	0.3	0.8	5.6	3.0	1.1	3.9	0.6	1.8	5.6	1.8
Yaoundé	2.6	0.7	1.0	5.8	3.7	0.5	4.2	1.5	1.5	4.7	1.1
Littoral (sans Douala)	9.1	2.2	2.1	14.2	5.7	13.7	17.2	13.1	15.2	19.3	11.9
Sud-Ouest	12.9	3.1	4.6	19.4	6.8	22.8	27.7	17.3	19.0	28.2	15.8
Sud	13.1	2.5	2.1	17.1	11.7	16.7	29.5	19.7	23.0	30.7	15.4
Ouest	14.6	4.9	5.7	20.9	6.1	25.8	30.5	26.4	21.2	34.8	12.3
Nord-Ouest	16.7	4.1	6.3	19.6	8.6	32.7	32.6	25.1	31.4	38.8	24.4
Centre (sans Yaoundé)	17.9	2.4	6.1	28.1	11.5	29.7	36.4	22.4	29.6	41.6	17.8
Adamaoua	26.9	24.1	20.5	27.9	26.8	37.1	25.1	24.4	26.6	50.1	23.1
Est	27.4	13.6	13.0	26.9	22.1	39.6	50.8	43.9	40.5	56.0	35.6
Nord	45.8	28.8	38.4	51.3	33.4	66.8	66.0	52.1	65.1	77.0	42.1
Extrême-Nord	54.0	51.9	52.1	48.3	42.5	79.2	53.8	54.1	70.8	86.4	43.1

Table 2: M-gamma Measures for Cameroon and Its Regions

	M_0^0	M_0^1	M_0^2	A	A ²
Cameroon	46.0	24.8	14.7	53.8	32.0
Douala	6.7	2.4	0.9	36.4	13.4
Yaoundé	6.5	2.6	1.1	40.3	16.6
Littoral (sans Douala)	19.9	9.1	4.4	45.5	22.3
Sud-Ouest	29.3	12.9	6.0	44.0	20.4
Sud	30.8	13.1	5.8	42.4	18.9
Ouest	35.1	14.6	6.4	41.8	18.2
Nord-Ouest	38.8	16.7	7.5	43.1	19.4
Centre (sans Yaoundé)	41.8	17.9	8.1	42.7	19.3
Adamaoua	51.1	26.9	15.4	52.7	30.1
Est	56.1	27.4	14.7	48.9	26.1
Nord	77.3	45.8	29.5	59.3	38.2
Extrême-Nord	86.7	54.0	36.2	62.3	41.8

Table 3: Inequality among the Multidimensionally Poor

Inequality measure	Cameroon
Gini	0.1780
Variance	0.2681
Atkinson geometric	0.0475
Atkinson harmonic	0.0903
Theil 0	0.0487
Theil 1	0.0492
Generalized Entropy 2	0.0514

Table 4: Percentage of Poor Experiencing Different Intensity Bands

	33–39%	40–49%	50–59%	60–69%	70–79%	80–89%	90–100%
Yaoundé	69%	12%	19%	0%	0%	0%	0%
Douala	82%	18%	0%	0%	0%	0%	0%
Littoral	44%	18%	22%	11%	0%	4%	0%
Sud-Ouest	49%	21%	18%	12%	0%	0%	0%
Sud	57%	20%	17%	4%	3%	0%	0%
Ouest	62%	14%	17%	7%	0%	0%	0%
Nord-Ouest	47%	23%	23%	7%	0%	0%	0%
Centre	55%	20%	16%	7%	0%	2%	0%
Adamaoua	26%	22%	21%	16%	6%	7%	1%
Est	39%	9%	28%	16%	4%	2%	1%
Nord	16%	15%	20%	25%	11%	9%	3%
Extrême-Nord	14%	12%	15%	24%	19%	9%	5%

7. Conclusions

In a recent paper, Alkire and Foster (2011a) presented a dual-cutoff approach to identifying the poor that prioritizes the most multiply deprived and an adjusted headcount ratio poverty measure that gauges the average the breadth of poverty across society. The methodology is now commonly being employed in national and international evaluations of multidimensional poverty. There are three properties of this methodology that have helped make it useful in practice: Subgroup Decomposability, which relates subgroup to overall poverty levels; Dimensional Breakdown, which relates multidimensional poverty levels to dimensional components; and Ordinality, which allows meaningful evaluations of poverty when variables are ordinal. The method also satisfies an array of invariance, subgroup, and dominance axioms, including two weak distribution-sensitivity properties that require multidimensional poverty not to fall in response to an increase in inequality among the poor.

This paper proposed a new Dimensional Transfer property that requires poverty to rise in response to a certain type of inequality-increasing transfer among the poor. Neither the headcount ratio nor the adjusted headcount ratio satisfies this property. However, following a suggestion in Alkire and Foster (2011a), we constructed an M -gamma class of poverty measures that builds upon the adjusted headcount ratio and has a subclass that satisfies the Dimensional Transfer axiom. We then devised an intuitive transformation from unidimensional to multidimensional poverty measures by converting the multidimensional achievement matrix into an attainment count distribution and applied a unidimensional poverty measure. We emphasized that the transformation provides a link between unidimensional and multidimensional *properties* as well; in particular, if the unidimensional poverty measure satisfies the traditional Monotonicity and Transfer properties then the multidimensional poverty measure satisfies Dimensional Monotonicity and Dimensional Transfer. The M -gamma measures can be constructed using a version of the FGT measures and inherit favorable properties from them. We concluded that it is a straightforward exercise to construct examples of multidimensional measures that conform to the strict distributional requirements of the Dimensional Transfer axiom.

We then observed that none of the measures found to satisfy Dimensional Transfer also satisfy Dimensional Breakdown, and this in turn led us to prove an impossibility result identifying a fundamental conflict between the two properties. Given this conclusion, and the importance of Dimensional Breakdown for poverty analysis, we recommended the use of M_0 as the base measure, augmented as necessary by additional measures, such as the squared count measure from the M -gamma class, to provide information about inequality among the poor. We illustrated our approach using 2011

data on the global MPI from Cameroon, a country where dimensional components of poverty vary considerably across regions and so Dimensional Breakdown is especially important. To gain additional information on inequality, we reported the three main M -gamma measures – the headcount ratio, the adjusted headcount ratio, and the squared count measure – and noted the similarity to standard presentations of the FGT measures. We observed where variations in intensity A cause re-ranking between H and M_0 , and where differences in the inequality-sensitive intensity A^2 cause re-rankings between M_0 and the squared count index M_0^2 . With the help of the attainment count distribution and the transformation, measuring multidimensional poverty can be as easy as using a unidimensional poverty measure.

Several of the insights described in this paper might be relevant for future research. The fact that multidimensional poverty can be reinterpreted as unidimensional poverty applied to an attainment count distribution immediately suggests a number of theoretical questions that might be explored. For example, the optimal budgeting exercises of Kanbur (1987a,b) and Bourguignon and Fields (1990) for the FGT measure might be reconsidered in the multidimensional context with, say, an M -gamma measure as the objective to minimize and separate pools of dimensional resources. How might the conclusions of Bourguignon and Fields (1990) – which assume perfect information and policy levers – be altered in the multidimensional context? The headcount ratio would likely still emphasize those whose poverty can most inexpensively be alleviated; the adjusted headcount ratio may well need to balance the benefits from reducing the breadth of poverty and the number of the poor; the squared count measure could shift the balance towards reducing the breadth of the poorest. In any case, the underlying multidimensionality clearly adds an additional layer of choices – the specific dimensions targeted for each person – that would require careful analysis in order to construct optimal policies. In contrast, what happens when information and policy levers are limited to subgroups of the population? The unidimensional results of Kanbur (1987a,b) suggest that when the squared count measure M_0^2 is the objective, it could be appropriate to target first the subgroup with the highest adjusted headcount ratio M_0^1 . Once again, multidimensionality would likely alter the optimal policy, with Dimensional Breakdown playing a pivotal role in its derivation. We should also note the potential benefit from applying other distributional measures to the entire attainment count distribution, thereby obtaining insight on, say, multidimensional inequality or mobility, when data are not cardinal. It would be interesting to explore these and other related questions.

We have emphasized the importance of Distributional Breakdown and have shown how the specific breakdown formula for the adjusted headcount ratio is used in practice. The space in between the general property and the specific formula for M_0 is a potential area for research. In particular, how the

weights and component functions might vary for different poverty status vectors is especially relevant. Note that since the weights are assumed to be positive, it is easy to adjust component functions to have the same weights for all domains and, indeed, to make them consistent with the identification step. As for the component functions, the main question is their salience. The censored headcount ratios from the adjusted headcount ratio are independently meaningful as the percentage of the population both poor and deprived in the given dimensions – a meaning that is preserved across domains. It would be interesting to understand what kinds of component functions are desirable and to find additional properties that might select the censored headcounts ratios.

Appendix

Proof of Proposition 3: Let \mathcal{M}_p be the attainment count methodology generated by a unidimensional methodology \mathcal{P} .

(i) Suppose that \mathcal{P} satisfies Weak Monotonicity, and let y' be obtained from y by an increment in a single achievement. Then either $a' = a$ if the increment does not remove a deprivation, or else a' is obtained from a by an increment in single attainment level, and hence $P(a'; \pi) \leq P(a; \pi)$ by Weak Monotonicity of P . Consequently $M_p(y'; z) \leq M_p(y; z)$, establishing Weak Monotonicity for \mathcal{M}_p .

(ii) Suppose that \mathcal{P} satisfies Monotonicity, and let y' be obtained from y by a decrement in a single achievement. Then either $a' = a$ if the decrement does not create a deprivation, or else a' is obtained from a by a decrement in single attainment level, and hence $P(a'; \pi) > P(a; \pi)$ by Monotonicity of P . Consequently $M_p(y'; z) > M_p(y; z)$, establishing Dimensional Monotonicity for \mathcal{M}_p .

(iii) Suppose that \mathcal{P} satisfies Weak Transfer, and let y' be obtained from y by an association-decreasing rearrangement among the poor involving persons h and i , for which $y_i > y_h$ while neither $y'_i > y'_h$ nor $y'_h > y'_i$. Then either $a' = a$ if the rearrangement does not switch a deprivation, or else a' is obtained from a by a progressive transfer from i to h , and hence $P(a'; \pi) \leq P(a; \pi)$ by Weak Transfer for P . Consequently $M_p(y'; z) \geq M_p(y; z)$, establishing Weak Transfer for \mathcal{M}_p .

(iv) Suppose that \mathcal{P} satisfies Transfer, and let y' be obtained from y by a dimensional rearrangement among the poor involving persons h and i , for which $y_i > y_h$ while neither $y'_i > y'_h$ nor $y'_h > y'_i$, and also $g_h^0 > g_i^0$ while neither $g_i^{0'} > g_h^{0'}$ nor $g_h^{0'} > g_i^{0'}$. Then the rearrangement involves a deprivation being “transferred” from h to i and, hence, a' is obtained from a by a progressive transfer from i to h . Consequently, $P(a'; \pi) < P(a; \pi)$ by Transfer for P and so $M_p(y'; z) > M_p(y; z)$, establishing Dimensional Transfer for \mathcal{M}_p . ■

References

- Alkire, S., and Foster, J. (2011a). Counting and Multidimensional Poverty Measurement. *Journal of Public Economics*, 95(7–8), 476–487.
- Alkire, S., and Foster, J. (2011b). Understandings and Misunderstandings of Multidimensional Poverty Measurement. *Journal of Economic Inequality*, 9(2), 289–314.
- Alkire, S., Meinzen-Dick, R., Peterman, A., Quisumbing, A. R., Seymour, G., and Vaz, A. (2013). The Women’s Empowerment in Agriculture Index. *World Development*, 53, 71–91.
- Alkire, S., and Robles, G. (2015). Multidimensional Poverty Index 2015: Brief Methodological Note and Results. *OPHI Policy Briefing* 31, University of Oxford.
- Alkire, S., Roche, J.M., and Vaz, A. (2015). Changes over Time in Multidimensional Poverty: Methodology and Results for 34 Countries. *OPHI Working Paper* 76, University of Oxford.
- Alkire, S., and Santos, M. E. (2010). Acute Multidimensional Poverty: A New Index for Developing Countries. *OPHI Working Paper* 38, University of Oxford.
- Alkire, S., and Santos, M.E. (2014). Measuring Acute Poverty in the Developing World: Robustness and Scope of the Multidimensional Poverty Index. *World Development*, 59, 251–274.
- Alkire, S., Foster, J., Seth, S., Santos, M.E., Roche, J.M., and Ballon, P. (2015). *Multidimensional Poverty Measurement and Analysis: A Counting Approach*. Oxford University Press.
- Angulo, R. (2016). From Multidimensional Poverty Measurement to Multisector Public Policy for Poverty Reduction: Lessons from the Colombian Case. *OPHI Working Paper* 102, University of Oxford.
- Atkinson, A. B. (1987). On the Measurement of Poverty. *Econometrica*, 55(4), 749–764.
- Atkinson, A. B. (2003). Multidimensional Deprivation: Contrasting Social Welfare and Counting Approaches. *Journal of Economic Inequality*, 1(1), 51–65.
- Atkinson, A. B., and Bourguignon, F. (1982). The Comparison of Multi-Dimensional Distribution of Economic Status. *The Review of Economic Studies*, 49, 183–201.
- Castillo Añazco, R., and Jácome Pérez, F. (2016). Medición de la Pobreza Multidimensional en Ecuador. Instituto Nacional de Estadística y Censos (INEC) [available at: http://www.ecuadorencifras.gob.ec/documentos/web-inec/Sitios/Pobreza_Multidimensional/ accessed on 22 Feb 2016].
- Bossert, W., Chakravarty, S. and D’Ambrosio, C. (2013). Multidimensional Poverty and Material Deprivation with Discrete Data. *Review of Income and Wealth*, 59(1), 29–43.
- Bourguignon, F., and Chakravarty, S. R. (2003). The Measurement of Multidimensional Poverty. *Journal of Economic Inequality*, 1(1), 25–49.
- Chakravarty, S. R., Mukherjee, D., Renade, R. R. (1998). On the Family of Subgroup and Factor Decomposable Measures of Multidimensional Poverty. *Research on Economic Inequality*, 8, 175–194.
- Chakravarty, S. R., and D’Ambrosio, C. (2006). The Measurement of Social Exclusion. *Review of Income and Wealth*, 53(3), 377–398.

- Chakravarty, S. R. and Silber, J. (2008). Measuring Multidimensional Poverty: The Axiomatic Approach. In N. Kakwani and J. Silber (Eds.), *Quantitative Approaches to Multidimensional Poverty Measurement* (pp. 192–209). Basingstoke: Macmillan.
- Chakravarty, S. R. (2009). *Inequality, Polarization and Poverty: Advances in Distributional Analysis*, vol. 6. New York: Springer.
- Clark, D.A., Hemming, R., and Ulph, D. (1981). On Indices for the Measurement of Poverty. *The Economic Journal*, 91(5), 515–26.
- CONEVAL (2009). *Methodology for Multidimensional Measurement of Poverty in Mexico*. México DF: CONEVAL.
- CONEVAL (2010). *Report on Multidimensional Poverty in México, 2008*. México DF: CONEVAL.
- CONPES (Consejo Nacional de Política Económica y Social) (2012). Official Methodology and Institutional Arrangements for the Measurement of Poverty in Colombia. *Documento Conpes Social*, 150. Bogotá, 28 May, in Spanish [available at colaboracion.dnp.gov.co/CDT/Conpes/150.pdf].
- Dhongde, S., Li, Y., Pattanaik, P.K., and Xu, Y. (2015). Binary Data, Hierarchy of Attributes, and Multidimensional Deprivation. *Mimeo*.
- Donaldson, D. and Weymark, J.A. (1986). Properties of Fixed-population Poverty Indices. *International Economic Review*, 27(3), 667 – 688.
- Foster, J., Greer, J., and Thorbecke, E. (1984). A Class of Decomposable Poverty Measures. *Econometrica*, 52(3), 761–766.
- Foster, J., Seth, S., Lokshin, M., and Sajaia, Z. (2013). *A Unified Approach to Measuring Poverty and Inequality: Theory and Practice*. Washington, DC: The World Bank.
- Foster, J., and Sen, A. K. (1997). On Economic Inequality after a Quarter Century, an annex to A. Sen, *On Economic Inequality*. Oxford: Oxford University Press, 107–220.
- Foster, J., and Shorrocks, A. F. (1991). Subgroup Consistent Poverty Indices. *Econometrica*, 59(3), 687–709.
- Foster, J., and Smith, S. (2016). Measuring Ultrapoverty: A New Class of Indices and an Application to Rural Ethiopia Panel Data. *Mimeo*. George Washington University.
- Gobierno de Chile Ministerio de Desarrollo Social (2015). [Nueva Metodología de Medición de Pobreza por Ingresos y Multidimensional](#), *Documentos Metodológicos* 28, Santiago: Gobierno de Chile, 26 January.
- Govt. of Costa Rica INEC (2015). *Methodology of the Costa Rican Multidimensional Poverty Index (MPI)*. October, 2015. In Spanish. [Available at www.inec.go.cr/anda4/index.php/catalog/153].
- Jayaraj, D., and Subramanian, S. (2009). A Chakravarty-D'Ambrosio View of Multidimensional Deprivation: Some Estimates for India. *Economic and Political Weekly*, 45(6), 53–65.
- Kanbur, S.M.R. (1987a). Measurement and Alleviation of Poverty, with an Application to the Effects of Macroeconomic Adjustment. *International Monetary Fund Staff Papers* 34(1), 60–85.
- Kanbur, S.M.R. (1987b). Transfers, Targeting and Poverty. *Economic Policy* 2(4), 112–147.
- Kolm, S. (1977). Multidimensional Egalitarianisms. *Quarterly Journal of Economics*, 91(1), 1–13.
- Maasoumi, E., and Lugo, M. A. (2008). The Information Basis of Multivariate Poverty Assessments. In N. Kakwani and J. Silber (Eds.), *Quantitative Approaches to Multidimensional Poverty Measurement* (pp. 1–29). New York: Palgrave-MacMillan.

- Pattanaik, P., Reddy, S., and Xu, Y. (2012). On Measuring Deprivation and Living Standards of Societies in a Multi-Attribute Framework. *Oxford Economic Papers*, 64, 43–56.
- Rippin, N. (2013) *Considerations of Efficiency and Distributive Justice in Multidimensional Poverty Measurement*. Doctoral Thesis, Georg-August-Universität, Göttingen.
- Rippin, N. (2015): Multidimensional Poverty in Germany: A Capability Approach. *Forum for Social Economics*. Published online.
- Royal Government of Bhutan Gross National Happiness Commission (2011). Turning Vision into Reality: The Development Challenges Facing Bhutan. *Eleventh Roundtable Meeting*. Thimphu: GNHC.
- Royal Government of Bhutan National Statistics Bureau (2012). *Bhutan Multidimensional Poverty Index*, Thimphu: NSB [www.nsb.gov.bt/publication/download.php?id=279 accessed on 9 Aug 2014].
- Sen, A. (1973). *On Economic Inequality*. New York: Oxford University Press.
- Sen, A. (1976). Poverty: An Ordinal Approach to Measurement. *Econometrica*, 44(2), 219–231.
- Sen, A. (1997). *On Economic Inequality. Expanded Edition with a Substantial Annex by James E. Foster and Amartya Sen*. New Delhi: Oxford University Press.
- Seth, S., and Alkire, S. (2013). Measuring and Decomposing Inequality among the Multidimensionally Poor using Ordinal Variables: A Counting Approach. *OPHI Working Paper 68*, University of Oxford.
- Shorrocks, Anthony F. (1995). Revisiting the Sen Poverty Index. *Econometrica*, 63(5), 1225–30.
- Stevens, S. S. (1946), On the Theory of Scales of Measurement. *Science*, New Series 103 (2684), 677–680.
- Thon, D. (1979). On Measuring Poverty. *Review of Income and Wealth*, 25, 429–440.
- Tsui, K. (2002). Multidimensional Poverty Indices. *Social Choice and Welfare*, 19(1), 69–93.
- Ura, K., Alkire, S., Zangmo, T., and Wangdi, K. (2012). *An Extensive Analysis of the Gross National Happiness Index*. Centre of Bhutan Studies, Royal Government of Bhutan. Available on www.grossnationalhappiness.com.
- Ura, K., Alkire, S., Zangmo, T., and Wangdi, K. (2015). *A Preliminary Analysis of the 2015 Gross National Happiness Index*. Centre of Bhutan Studies and GNH Research, Royal Government of Bhutan. Available on www.grossnationalhappiness.com.
- Watts, H.W. (1968). An Economic Definition of Poverty. In Moynihan, D.P. (Ed.), *On Understanding Poverty*, pp. 316–329. New York: Basic Books.