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Normative Choices and Tradeoffs when Measuring Poverty over Time

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Abstract

This paper examines the aggregation of an indicator of wellbeing over time and across people to measure poverty. We characterise the general form of an intertemporal poverty measure under mild normative principles and show that it must embody an unambiguous ordering of possible trajectories of an individual's wellbeing. We motivate further normative principles and examine their consequences for the form of the measure, showing that some measures suggested in the literature are not consistent with these principles. We discuss additional stronger properties that may be argued to be desirable for an intertemporal or chronic poverty measure. We identify compatibilities and tradeoffs among certain of these properties. For example, a poverty measure cannot simultaneously capture chronicity of poverty and sensitivity to fluctuations. We argue that a poverty analyst should choose among these properties according to context and the particular conception of poverty she seeks to measure.

Keywords: Intertemporal welfare, Poverty measurement

JEL classification: D63, D91, I32

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1 Introduction

Incorporating the dimension of time into poverty measurement is a crucial and complicated task. In economics, efforts are in their infancy to extend what is a fairly strong consensus on the principles of poverty measurement for a single time period to the case of multiple time periods. Several measures, which have very different structures and properties, have been proposed in this new literature.

The motivation to get this right is clear: both economists and non-economists studying the lives of the poor have highlighted the shortcomings of unidimensional poverty measurement, given that poverty is a multidimensional state characterised by a lack of assets, feelings of insecurity or vulnerability, and deprivation in many dimensions and over extended durations.[1] Whilst much research has agreed in principle with this critique, in practice many challenges arise when attempting to be true to the broader definition of poverty using economic tools and the available empirical data.

The principles of measuring poverty over time however are not evident. Given information on wellbeing in multiple time periods, how should we aggregate each individual's poverty experience and rank them? This paper addresses such questions in a formal manner, adding to a fledgling literature on intertemporal poverty.

An influential paper by Rodgers and Rodgers (1993) using US data introduced conceptual thinking on chronic poverty that has led to efforts (mainly in the context of developing countries) to find a composite measure of intertemporal poverty.[2] One of the most-cited approaches has been proposed by Jalan and Ravallion (2000), essentially an empirical application of the Rodgers and Rodgers approach. Both involve decomposing an FGT-style[3] intertemporal poverty measure into 'chronic' and 'transient' components, where the chronic component – which has been most widely adopted and applied in the empirical literature – focuses on those whose *average* income or consumption is below the poverty line during the period under scrutiny.

One critique of this averaging approach to chronic poverty measurement is that it implicitly assumes perfect substitution of wellbeing between time periods, both above and below the poverty line. Several other authors have proposed methods to characterise poverty over time, which are generally termed 'chronic poverty' measures and are rooted in the idea of 'spells' (see Bane and Ellwood (1986)). For example, this is the approach used in measures proposed by Foster (2009) and Calvo and Dercon (2009). These measures allow some degree of substitution between wellbeings in different periods provided they are all below the poverty line. However, they allow no substitution at all across the poverty line, which we argue may be normatively undesirable. Gradin, Del Rio, and Canto (2011) generalise an approach pioneered by Bossert, Chakravarty, and D'Ambrosio (2012) that penalises contiguous episodes of poverty to reflect the idea that time spent continuously in poverty may have an impact that is more than additive. These measures are sensitive to 'chronicity' of poverty in a way that earlier measures are not. All of the above measures are separable over people but not, in general, over time (i.e. it is the lifetime trajectory of each individual that matters). Hoy and Zheng (2007) take a completely different approach, imposing separability both over time and across people.

In this paper, we take a normative approach to the problem of how to evaluate poverty in profiles of information based on observations of many individuals at several points in time.

[1] For philosophical foundations see Nussbaum and Sen (1999) and the various writings of Amartya Sen e.g. (1997), (2001).

[2] As opposed to alternative approaches to conceptualising poverty over time such as the spells approach (See Bane and Ellwood (1986) for a seminal paper).

[3] Foster, Greer, and Thorbecke (1984)'s seminal paper introducing the Foster-Greer-Thorbecke (FGT) suite of poverty measures including the headcount, poverty gap and squared poverty gap.

Our first step, in section 2, is to characterise the general functional form of an intertemporal poverty measure under very mild normative assumptions. The main result is that the assumptions of representability, population symmetry (anonymity) and a basic consistency requirement over subsets of the population entail that the poverty measure must be additively separable across individuals. This result, a corollary of a theorem proved by Quinn (2009) is related to the main result of Foster and Shorrocks (1991) but is formulated in our framework of intertemporal poverty measurement and is more general, as the strong continuity assumptions invoked by Foster and Shorrocks are relaxed. The result is therefore sufficiently general to encompass all of the intertemporal poverty measures suggested in the recent literature, some of which have substantial discontinuities. It follows that the intertemporal poverty measure must induce an unambiguous ordering of the space of possible individual trajectories; this provides a framework in which we can explore further restrictions that arise from stronger normative assumptions.

In section 3, we outline some normative properties which we argue to be weak and uncontroversial in the context of intertemporal poverty measurement. We start with the trajectory ordering; monotonicity and focus properties are ubiquitous in the literature on poverty measurement and we invoke weak versions which we argue to be uncontroversial while returning to consider alternative stronger versions in section 4. We also motivate and invoke continuity of the ordering of the trajectory space, discussing why this is desirable in an intertemporal poverty measure despite the fact that several measures proposed in the recent literature are not continuous. Invoking these properties allows us to characterise restrictions on the form of the function representing the trajectory ordering; some but not all of the recently suggested intertemporal poverty measures fall into this class. We generalise the concept of a poverty line to the intertemporal context and observe that every possible trajectory must be regarded as representing equivalent intertemporal poverty as some constant-wellbeing trajectory. Section 3 concludes with a brief discussion of aggregation across individuals to represent the social ordering. As this subject has been examined thoroughly in the literature on cross-sectional or static poverty measurement, we refer to general results from that literature, explaining how they are applicable in the intertemporal context.

In section 4 we then explore several normative properties of the trajectory ordering that could be seen as desirable (and have been proposed in the literature). In some cases these are analogues in the intertemporal case of properties of the social ordering that are taken as axiomatic in the static poverty literature. However the analogy is not simple, and there are several options. We discuss at length the problem of how to evaluate two trajectories with the same average wellbeing when one is more volatile than another. If we are to compensate periods of low wellbeing, what is the relevant rate of intertemporal compensation or substitution? Another concern is whether a trajectory which, *ceteris paribus*, has more time spent below some 'poverty line' should be considered more poor. The final concern addressed more briefly is whether the order of time periods matters. No poverty measure has yet been introduced that relaxes time symmetry and conforms to continuity, though we are not able to rule out that this is possible. Our conclusions show that some of these clearly desirable properties are incompatible, or, if invoked, severely restrict the underlying functional form of the poverty function. Our analysis should enable those working in the field of intertemporal poverty measurement to clarify tradeoffs and compatibilities between different properties that have not yet been made explicit in the literature.

2 Framework and General Form of the Intertemporal Poverty Measure

In this section we establish the framework of analysis, which allows us to focus on the specific problems of intertemporal poverty measurement. We then state three mild normative assumptions and show that these assumptions characterise a general functional form for an intertemporal poverty measure in our framework. This functional form is shared by all intertemporal poverty measures suggested in the recent literature, so we infer that these measures all satisfy the three normative assumptions.

The general functional form enables us to distinguish clearly between the concept of an ordering of an individual's wellbeing trajectories and the concept of a social ordering. This facilitates our discussion, in subsequent sections, of the properties of the trajectory ordering and social ordering induced by an intertemporal poverty measure.

2.1 Information and Evaluation Framework

In order to explore the particular problems of intertemporal poverty measurement we assume that the poverty analyst has access to accurate observed or hypothetical information about the wellbeing of several people over an extended period of time. In this context, the simplest way to model an extended period of time is as a finite number of discrete time periods, $T = \{1, 2, \ldots, T\}$ for some $T \in \mathbb{N}$. This is relevant for practical application as it reflects the time structure of any panel or longitudinal data set.

We shall assume that the indicator of wellbeing is cardinally measurable with a level below which it cannot descend, thus an individual's wellbeing in any period may be represented by a non-negative real number multiplied by a unit. Cardinality of the indicator means that in *measurement* terms the unit has a meaningful interpretation, and for example, the difference between zero and one units is equivalent to the difference between 10 and 11. It *does not* follow that individual or social preferences need consider these differences to be equivalent; we return to this issue in section 3. Furthermore, the indicator of wellbeing is assumed to be interpersonally and intertemporally comparable so that meaningful comparisons may be made between different people at different points in time.[4]

Assumption 1 (Individual-Period Wellbeing)**.** *The wellbeing w_{it} of an individual labelled i in a time period labelled t is an element of the set $W = \{xu | x \in \mathbb{R}_+\}$ where u is the unit of measurement.*

Note that, given the unit of the individual wellbeing indicator u, the wellbeing of a particular individual i in *each* period $t = 1, 2, \ldots, T$ may be *represented* by the T-tuple $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iT}) \in \mathbb{R}_+^T$ where each $x_{it} = \frac{w_{it}}{u}$. We shall describe this T-tuple as i's **wellbeing trajectory**. The wellbeing trajectories of each of $n \in \mathbb{N}$ individuals may be represented by an

[4]Wellbeing is perhaps best conceptualised as a multidimensional latent variable for which we will never have a perfect measure; we must therefore must use a proxy. Ravallion (1994) and Grosh, Glewwe, and Bank. (2000) argue that, subject to many caveats, a discounted, equivalised imputed value of consumption is often the best available measure of individual-period wellbeing. Income data may also be used, but it should be treated with caution as income is frequently subject to interpersonal redistribution and intertemporal smoothing. Of course, the arguments made in the literature on multidimensional poverty measurement, for example Tsui (2002) and Bourguignon and Chakravarty (2003) will apply here, and so a better indicator may be some composite index. However, the introduction of multiple dimensions of wellbeing raises issues that are not dealt with in this paper, for example the order of intertemporal, multidimensional and social aggregation.

$(n \times T)$ **wellbeing profile**

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1T} \\ x_{21} & x_{22} & \cdots & x_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nT} \end{pmatrix} \tag{1}$$

in which the i^{th} row represents the i^{th} individual's wellbeing trajectory. Note that a column of the matrix may be interpreted as a single-period **distribution of wellbeing**, while the whole profile may be thought of as a **distribution of trajectories** of wellbeing. Given a wellbeing profile X, let $n(X) = n$ if $X \in \mathbb{R}^{nT}$; n is the number of rows, or number of individuals represented in the profile.

The poverty analyst may wish to compare poverty in observed and hypothetical populations of different sizes, so her method of evaluation of poverty must admit wellbeing profiles of varying dimension n. We shall require the poverty analyst's method of evaluation to satisfy basic consistency requirements, as is conventional in the literature on individual and social choice; any profile contains at least as much poverty as itself (reflexivity), while if profile X_1 contains at least as much poverty as profile X_2 and profile X_2 contains at least as much poverty as profile X_3 then profile X_1 contains at least as much poverty as profile X_3 (transitivity).

Assumption 2 (Poverty Ordering on a Set of Wellbeing Profiles)**.** *Given u and some integers $N \geq 3$ and $T \geq 3$, the poverty analyst's method of evaluation specifies a reflexive and transitive binary relation $\succsim$ (a partial preorder) on the space of wellbeing profiles*

$$\mathcal{X}_{NT} = \bigcup_{n=3}^{N} \mathbb{R}_{+}^{nT}.$$

The relation $\succsim$ is read 'contains at least as much poverty as'.

These two assumptions, which we maintain throughout this paper, provide a framework in which we shall carry out our analysis of intertemporal poverty measurement. This framework (in particular the information structure) is consistent with, or more general than, that assumed by most other studies in this field.[5] This means that our analysis and results may be directly compared with these other studies. The framework does not allow the poverty analyst to compare profiles of different durations,[6] containing incomplete or missing information, or imperfectly observed data, all of which are relevant in an applied context. However, any more general domain for the poverty ordering that permitted one or more of these possibilities would have a subsets of the form $\mathcal{X}_{NT}$. Given a particular set of normative principles, we would expect the poverty analyst's method of evaluation for a more general domain to be consistent with her evaluation over the restricted domain. It is, therefore, a valuable exercise to examine methods of evaluation for the restricted domain, which may in further work be extended to explore the further issues that arise when measuring poverty of a more general domain.

[5]Note that some authors, for example Bossert, Chakravarty, and D'Ambrosio (2012), take as primary a real-valued measure of individual *poverty* rather than wellbeing. Without any loss of generality this may be considered to be a non-increasing transformation of a real-valued measure of wellbeing and is thus consistent with our framework. Their approach is less general than ours and does impose some restrictions on the ordering of trajectories of wellbeing which, in the next section, we shall explore and argue to be stronger than can be normatively motivated.

[6]The information structure used by Bossert, Chakravarty, and D'Ambrosio (2012) allows for comparison of profiles including trajectories of different durations; we wish to focus on the important issues that arise in making fixed-duration comparisons before considering the more general case.

2.2 Elementary Normative Assumptions

We now introduce three mild normative assumptions that allow us to characterise the general form of an intertemporal poverty measure representing the poverty analyst's method of evaluation and thus in subsequent sections to focus on the specific issues of poverty measurement in an intertemporal context.

First, in order to ensure representability of the poverty ordering $\succsim$ by a real-valued poverty measure we must specify that the binary relation is complete, so that the poverty analyst can order any pair of wellbeing profiles. Furthermore it must not be pathologically discontinuous. As is well known, the lexicographic ordering on an uncountable product space has an uncountable set of discontinuities and may not be represented by a real-valued function (Debreu, 1954). However, full continuity is not required for representability and may not be desirable in the context of poverty measurement; the headcount measure, for example, is not continuous. Debreu (1954) gives necessary and sufficient conditions for representability that involve a minimal restriction to discontinuity of the ordering (see also Jaffray (1975)).[7]

Assumption 3 (Representability)**.** *Given assumptions 1 and 2, the poverty ordering $\succsim$ on $\mathcal{X}_{NT}$ satisfies the following properties:*

1. *$\succsim$ is complete; $X \succsim Y$ or $Y \succsim X$ for all $X, Y \in \mathcal{X}_{NT}$.*
2. *The preorder topology induced by $\succsim$ on $\mathcal{X}_{NT}$ is second countable; it has a countable basis.*

Second, it is conventional in welfare analysis and poverty measurement to treat individuals anonymously or symmetrically; in the context of static poverty measurement this principle is stated explicitly by, for example, Chakravarty (1983) and Foster and Shorrocks (1991). The poverty analyst should evaluate as equally poor profiles in which the same distribution of wellbeing trajectories is allocated differently among the individuals.[8] This assumption will ensure that the poverty measure is a symmetric function of all individuals' trajectories of wellbeing.[9]

Assumption 4 (Population Symmetry)**.** *Given assumptions 1 and 2, the poverty ordering $\succsim$ on $\mathcal{X}_{NT}$ satisfies the following property: For all wellbeing profiles $X, Y \in \mathcal{X}_{NT}$ such that $n(X) = n(Y)$ and $X = A_pY$ for some $(n(X) \times n(X))$ permutation matrix[10] A_p, $X \sim Y$.*

The third assumption is another consistency requirement. Consider a subset of the population containing a fixed number of individuals and the complement of that subset (the rest of the population). Consider varying the profile of wellbeings for the subset while holding the number of individuals in the complement and their wellbeing trajectories fixed. The poverty analyst's ordering of wellbeing profiles for the whole population should be independent of the (fixed) size

[7] We cannot conceive of any situation in which the poverty analyst would not be happy to accept this minimal restriction to discontinuity of the ordering. In fact, in subsequent sections we argue for a much stronger degree of continuity for the poverty ordering. Nonetheless, we retain the weakest possible restriction for now in order to maximise the generality of the results in this section.

[8] It follows immediately that the wellbeing indicator is the only information pertaining to each individual that the poverty analyst may take into account. We have already noted that the indicator should be interpersonally comparable; it must be carefully defined and measured, with appropriate equivalisation, to take account of all relevant individual characteristics and thus permit interpersonal comparisons.

[9] Note that Hoy and Zheng (2007) effectively impose an assumption of invariance under permutations of time periods as well as individuals' identities due to their assumption of invariance with respect to the order of aggregation across individuals or time. This leads to a very strong restriction on the ordering of trajectories which, in the next section, we shall argue is inappropriate in the context of intertemporal poverty measurement.

[10] A square matrix of 1s and 0s with exactly one 1 in each row and column.

of the complement and the (fixed) profile of wellbeings experienced by the individuals in the complement.

This may be motivated further by considering what the poverty ordering represents, in the case that the profile for some subset (the 'complement') is fixed. The poverty ordering must represent the poverty analyst's ordinal evaluation of the degree of poverty in the varying-profile subset only. If our assumption above fails to hold then we must have a perverse scenario in which the poverty analyst's ordering of a pair of profiles for the subset *reverses* when the wellbeing profile for the complement changes. We cannot conceive of any circumstances in which this seems reasonable.

This assumption is new in the literature but is closely related to the *subgroup consistency* property introduced to poverty measurement by Foster and Shorrocks (1991). Like Foster and Shorrocks[11] we are able to use our assumption to show that the poverty measure is additively separable over individuals and thus, in this context, defines a (representable) ordering of trajectories. It differs slightly from *subgroup consistency* in that the framework is generalised to multiple time periods (we consider trajectories of wellbeings rather than just an instantaneous measure) and the assumption is expressed in terms of the poverty ordering rather than a real-valued poverty measure. It differs more substantially in two ways; it is *weaker* in that it requires the profile of wellbeing trajectories for the complementary subset to be fixed, and not just the measure of poverty for that complementary profile, while it is *stronger*[12] in that we require the ordering of profiles for the whole population to remain unchanged regardless of the *size* of the fixed-profile complementary subset. We consider that this strengthening is no more difficult to motivate than the weaker form, as its failure would imply existence of a similar perversity in the poverty ordering.

In order to define this assumption we need to introduce a little more notation.[13] In assumption 2 the space of wellbeing profiles $\mathcal{X}_{NT} = \bigcup_{n=3}^{N} \mathbb{R}_+^{nT}$ was introduced. Note that $\mathbb{R}_+^{nT} = (\mathbb{R}_+^T)^n = \prod_{i=1}^{n} \mathbb{R}_+^T$. Let $S \subset \{1, 2, \ldots, n\}$ and let $\bar{S} = \{1, 2, \ldots, n\} \setminus S$. Then $\mathbb{R}_+^{nT} = \prod_{i \in (S \cup \bar{S})} \mathbb{R}_+^T = \prod_{i \in S} \mathbb{R}_+^T \times \prod_{i \in \bar{S}} \mathbb{R}_+^T$. Consider some profile $X = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \in \mathbb{R}_+^{nT}$. Let $X_S = \{\boldsymbol{x}_i | i \in S\}$ and let $X_{\bar{S}} = \{\boldsymbol{x}_i | i \in \bar{S}\}$.

Assumption 5 (Variable-Population Subset Consistency)**.** *Given assumptions 1 and 2, the poverty ordering* $\succsim$ *on* $\mathcal{X}_{NT}$ *satisfies the following property: Given integers* $T \geq 3$ *and* $N \geq 3$, *for all* $n, n' \in \{3, \ldots, N\}$ *and all wellbeing profiles* $X, Y \in \mathbb{R}_+^{nT} \subset \mathcal{X}_{NT}$ *and* $X', Y' \in \mathbb{R}_+^{n'T} \subset \mathcal{X}_{NT}$ *such that for some* $S \subset \{1, 2, \ldots, n\} \cap \{1, 2, \ldots, n'\}$, $\bar{S} = \{1, 2, \ldots, n\} \setminus S$ *and* $\bar{S}' = \{1, 2, \ldots, n'\} \setminus S$, $X_S = X'_S$, $Y_S = Y'_S$, $X_{\bar{S}} = Y_{\bar{S}}$ *and* $X'_{\bar{S}'} = Y'_{\bar{S}'}$, $X \succsim Y \Leftrightarrow X' \succsim Y'$.

2.3 General Form of the Poverty Measure

The mild normative assumptions (3, 4 and 5) introduced above allow us to characterise a general intertemporal poverty measure which, given n, is additively separable across individuals. We may see immediately from the functional form of this general poverty measure that under these

[11] Their main result, that this entails additive separability across individuals, is based on results due to Gorman (1968) and Debreu (1960) linking order separability with additive separability of a representing function. These results rely on strong topological assumptions which Quinn (2009) relaxes. We invoke one of her results below.

[12] This enables us to show that the trajectory ordering is well defined without needing to invoke *replication invariance* at this stage, as Foster and Shorrocks do.

[13] We use the notation $\bar{S}$ to denote the complement of a set S rather than the more usual S', as we are using $'$ with another meaning. Our notation is similar to Gorman (1968) in his related analysis of additively separable representation of separable orderings.

assumptions, the poverty analyst's method of evaluation induces an unambiguous ordering of the space of wellbeing trajectories $\mathbb{R}^T_+$.

This result, a corollary of a theorem proved by Quinn (2009) is related to the main result of Foster and Shorrocks (1991) and similar results due to Blackorby, Bossert, and Donaldson (2005) but is formulated in our framework of intertemporal poverty measurement and is more general, as are the strong topological assumptions invoked by these authors. The result is therefore sufficiently general to apply to all of the intertemporal poverty measures suggested in the recent literature, some of which induce complex discontinuities in the ordering of the trajectory space.

Proposition 1. *Given assumptions 1 and 2, the poverty ordering $\succsim$ on $\mathcal{X}_{NT}$ satisfies the normative assumptions 3, 4 and 5 if and only if there exists a function $\pi : \mathbb{R}^T_+ \to \mathbb{R}$ and for each $n \in \{3, \ldots, N\}$ a strictly increasing function $G_n : \mathbb{R} \to \mathbb{R}$ such that*

$$P : X \mapsto G_{n(X)} \left(\sum_{i=1}^{n(X)} \pi(x_{i1}, x_{i2}, \ldots, x_{iT}) \right) \tag{2}$$

represents $\succsim$ on $\mathcal{X}_{NT}$

Proof. This result is a direct corollary of Quinn (2009), theorem 3.4. Our assumptions 1 and 2 are sufficient (but not necessary) for the presuppositions of Quinn (2009), theorem 3.4. (The result is therefore a special case of that theorem.) It remains to observe that our remaining assumptions 3, 4 and 5 are exactly equivalent to the properties invoked in Quinn (2009), theorem 3.4 on the restricted domain characterised by assumptions 1 and 2; the result therefore holds. □

We may observe that any intertemporal poverty measure of the general form (2) induces an unambiguous ordering of the space of wellbeing trajectories $\mathbb{R}^T_+$, represented by the function $\pi : \mathbb{R}^T_+ \to \mathbb{R}$. Let us denote this ordering $\succsim_T$. Without any loss of generality we may decompose this function $\pi = g \circ p$ where $g : \mathbb{R} \to \mathbb{R}$ is a strictly increasing transformation and $p : \mathbb{R}^T_+ \to \mathbb{R}$ represents $\succsim_T$ on $\mathbb{R}^T_+$. This gives us the equivalent form for our general intertemporal poverty measure,

$$P : X \mapsto G_{n(X)} \left(\sum_{i=1}^{n(X)} g(p(x_{i1}, x_{i2}, \ldots, x_{iT})) \right). \tag{3}$$

This decomposition will allow us, in the subsequent sections, to focus on the normative issues that arise when measuring poverty in an intertemporal context by examining separately

- the properties of the ordering of the trajectory space $\succsim_T$, represented by p,
- the properties of the fixed-n social ordering insofar as they are not inherited from $\succsim_T$, represented by $\sum g(.)$, and
- the properties of the variable-n social ordering insofar as they are not already inherited from the fixed-n social ordering, represented by the functions G_n.

All of the intertemporal poverty measures suggested in the recent literature may be expressed as functions of the form (3). Some authors, for example Gradin, Del Rio, and Canto (2011), have simply assumed this structure; Bossert, Chakravarty, and D'Ambrosio (2012) axiomatize the additive-across-individuals structure having *assumed* a two-step procedure of intertemporal aggregation followed by aggregation across individuals. Calvo and Dercon (2009) discuss alternative possibilities (varying the order of additive aggregation over time, across individuals and a transformation step) without rigorously relating the alternative approaches to the normative properties of the measure.[14]

Our Proposition 1 shows that necessary and sufficient conditions for the poverty measure to have this functional form are that it satisfies our assumptions of representability, population symmetry and variable-population subset consistency. We may therefore infer that all of the poverty measures suggested in the literature do satisfy these three basic properties. Furthermore, the normative assumptions that are embodied by these measures may be analysed in the decomposition framework described above, which we believe will contribute to the clarity of this literature.

We conclude by noting that the normative assumptions imposed thus far are not specific to poverty measurement; there is no requirement that a population experiencing lower wellbeing is evaluated as 'more poor'. In the next section we introduce mild normative principles that are specific to intertemporal poverty measurement and analyse their impact on the poverty ordering and the form of the poverty measure.

3 Normative Principles for Intertemporal Poverty Measurement

We maintain throughout this section the assumptions introduced in section 2. The poverty analyst's method of evaluation may therefore be represented by a poverty measure of the form (3) and we may analyse sequentially properties of the trajectory ordering and properties of the social ordering.

We introduce in section 3.1 mild normative principles applicable to the trajectory ordering which we argue to be appropriate in the context of intertemporal poverty measurement. These are shown to restrict but not fully characterise the trajectory ordering. [15] The restriction is strong enough to allow us to proceed in section 3.2 to an analysis of the social ordering; we follow the established consensus in the literature on cross-sectional poverty measurement and are able to refine the form of the general intertemporal poverty measure.

3.1 The Trajectory Ordering

Throughout this subsection we consider properties of the trajectory ordering $\succsim_T$ on $\mathbb{R}^T_+$, represented by the function $p : \mathbb{R}^T_+ \to \mathbb{R}$. p may be interpreted as an (ordinal) **individual intertemporal poverty measure**. We restrict our discussion to properties that embody mild normative principles, which we argue *all* intertemporal poverty measures should reflect. These properties, *weak monotonicity*, *CW-focus*, *CW-strict monotonicity* and *Euclidean continuity* are

[14] In relation to Calvo and Dercon (2009), our Proposition 1 demonstrates that aggregating over time first is necessary under our basic normative assumptions. However the intertemporal aggregation need not be additive and transformations may be taken at any step. Hoy and Zheng (2007) require that their measure be invariant to the order of intertemporal and social aggregation and thus demonstrate that it must be additively separable in both dimensions; however, they do not give any normative motivation for this requirement and we assert that it may not be appropriate in the context of intertemporal poverty measurement.

[15] This is reasonable as we believe that there exist alternative and potentially equally valid approaches to the measurement of poverty over time, which would reflect alternative principles held by the poverty analyst. Alternative approaches which are consistent with our restrictions will be discussed in section 4.

shown to characterise a broad class of trajectory orderings; several measures proposed in the recent literature do not possess all of these properties and therefore do not fall into this class.

We start with a property that is fundamental to the concept of *poverty* and, in various forms, is ubiquitous in the cross-sectional, multidimensional and intertemporal poverty measurement literature. All other things being equal, an increase in an individual's wellbeing in any time period should not increase the individual's intertemporal poverty measure p. Equivalently, in the context of the trajectory ordering $\succsim_T$ on $\mathbb{R}_+^T$, a trajectory $\boldsymbol{x}$ which differs from a trajectory $\boldsymbol{y}$ only by having greater wellbeing in a single time period should be ordered as weakly less poor than $\boldsymbol{y}$.

Property 1 (Weak Monotonicity). *Given T, $\succsim_T$ satisfies* weak monotonicity *if, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^T$ such that $x_t > y_t$ for some $t \in \{1, 2, \ldots, T\}$ and $x_\tau = y_\tau$ for all $\tau \in \{1, 2, \ldots, T\} \setminus \{t\}$, $\boldsymbol{y} \succsim_T \boldsymbol{x}$.*

An individual intertemporal poverty measure p represents a trajectory ordering $\succsim_T$ which has the property of *weak monotonicity* if and only if it is a weakly decreasing function of the individual's wellbeing in each period.

Weak monotonicity is a weak property and does not ensure any sensitivity of the measure to lack of wellbeing; at least in some contexts the poverty analyst would expect the measure of poverty to increase when an individual's wellbeing decreases. One remedy might be to strengthen the property such that a trajectory must be evaluated to contain strictly more poverty whenever an individual's wellbeing in any period decreases. However, when aggregated across individuals this would yield a 'poverty measure' that was sensitive to the wellbeings of all individuals, even those who are consistently very well-off.

Following Sen (1976) it is conventional in the poverty measurement literature to solve this problem by distinguishing between *identification* of the poor among the population being studied and *aggregation* of information about only those identified as poor to construct the index of poverty. It follows that the index should not be sensitive to the level of wellbeing of those not identified as poor (the 'focus principle'). This is achieved in the case of static poverty measures by choosing a level of the wellbeing indicator, the **poverty line** $zu \in W$ where u is the unit of measurement of the wellbeing indicator, and requiring that the measure is not sensitive to wellbeings that lie above this line.[16]

It is not entirely straightforward to extend this concept to the intertemporal context with its multiple time periods; there is an analogous challenge in the multidimensional context.[17] In order to simplify the discussion we shall start by considering the space of **constant-wellbeing trajectories** $C^T = \{\boldsymbol{x} \in \mathbb{R}_+^T | x_t = x_1, t \in \{2, 3, \ldots, T\}\}$, a subset of the trajectory space $\mathbb{R}_+^T$. We may define a constant-wellbeing 'poverty line' $\boldsymbol{z} = (z, z, \ldots, z) \in C^T$ such that the trajectory ordering evaluates as equivalent any two constant-wellbeing trajectories in which the constant level of wellbeing is greater than z, but evaluates as strictly more poor any constant-wellbeing trajectory in which the constant level of wellbeing is less than z.

[16] If z is chosen independently of the observed distribution of wellbeings, this corresponds to the concept of absolute rather than relative poverty, an approach that we shall maintain throughout the paper. Note that identification could, in principle, be based on information different from that represented by the wellbeing indicator. We shall ignore this possibility; if such data were available and informative about poverty, we maintain that it should be incorporated in the wellbeing indicator.

[17] Bourguignon and Chakravarty (2003) discuss two alternative properties, *strong focus* and *weak focus*, whose analogues in the intertemporal context we discuss below. These are not the only two possibilities; we shall argue in the subsequent section for a focus property intermediate in strength between the two.

Property 2 (CW-Focus)**.** *Given* T*,* $\succsim_T$ *satisfies* CW-focus *if there is some* $z \in \mathbb{R}_+$ *such that for all* $\boldsymbol{x}, \boldsymbol{y} \in C^T$ *such that* $z < x_1 < y_1$*,* $\boldsymbol{x} \sim_T \boldsymbol{y}$*, and for all* $\boldsymbol{x}, \boldsymbol{y} \in C^T$ *such that* $x_1 < z < y_1$*,* $\boldsymbol{x} \succsim_T \boldsymbol{y}$*.*

Intuitively, any function p that represents $\succsim_T$ on $\mathbb{R}_+^T$ and in particular on C^T is not sensitive to changes in the (constant) wellbeing level above a critical value z. We must be careful not to regard this constant-wellbeing poverty line $\boldsymbol{z}$ as a direct intertemporal analogue of the static poverty line, as it is only one element of the trajectory space $\mathbb{R}_+^T$.

Informally, it may extend to a hypersurface of 'marginally poor' trajectories which is a better analogue of the poverty line; we shall formalise this concept later.

Returning to the whole trajectory space $\mathbb{R}_+^T$, we may now define the concept of poor and non-poor trajectories. Assuming that $\succsim_T$ satisfies *CW-focus*, a trajectory $\boldsymbol{x} \in \mathbb{R}_+^T$ is a **poor trajectory** if there exists some $\boldsymbol{y} \in C^T$ such that $y_1 > z$ and $\boldsymbol{x} \succ_T \boldsymbol{y}$. A trajectory $\boldsymbol{x} \in \mathbb{R}_+^T$ is a **non-poor trajectory** if there exists some $\boldsymbol{y} \in C^T$ such that $y_1 > z$ and $\boldsymbol{x} \sim_T \boldsymbol{y}$. It will be convenient to label the space of trajectories identified as poor; let $\Phi^T = \{\boldsymbol{x} \in \mathbb{R}_+^T | \exists \boldsymbol{y} \in C^T, y_1 > z, \boldsymbol{x} \succ_T \boldsymbol{y}\}$. Denote by Z^T that part of the topological boundary (in the Euclidean topology) of Φ^T which lies inside $\mathbb{R}_{++}^T$; this may be regarded as the intertemporal analogue of the poverty line. We do not yet know whether $\boldsymbol{z}$ is an element of Φ^T but we can show that it is an element of Z^T.

Note that *weak monotonicity* in conjunction with *CW-focus* entails a focus property analogous to the *weak focus* property suggested by Bourguignon and Chakravarty (2003) in the multidimensional context. Consider any $\boldsymbol{x} \in \mathbb{R}_+^T$ such that $x_t > z$ for each $t = 1, \ldots, T$; there exists some $\boldsymbol{y} \in C^T$ such that $x_t > y_1 > z$ for each $t = 1, \ldots, T$ and so by *weak monotonicity* $\boldsymbol{y} \succsim_T \boldsymbol{x}$. Similarly there exists some $\boldsymbol{y}' \in C^T$ such that $y_1' > x_t > z$ for each $t = 1, \ldots, T$ and so by weak monotonicity $\boldsymbol{x} \succsim_T \boldsymbol{y}'$. By *CW-focus* $\boldsymbol{y}' \sim_T \boldsymbol{y}$ and so by transitivity $\boldsymbol{x} \sim_T \boldsymbol{y}$; $\boldsymbol{x}$ is a non-poor trajectory. Applying *CW-focus* again, the trajectory ordering must evaluate $\boldsymbol{x}$ as equivalent to any trajectory of constant wellbeings greater than z.

Intuitively, under *weak monotonicity* and *CW-focus*, any function p that represents $\succsim_T$ is not sensitive to changes in the wellbeing in any period, for an individual whose level of wellbeing lies above the critical level z in *every* period. This property is satisfied by all of the intertemporal poverty measures suggested in the recent literature.

An alternative approach to focus is *strong focus*, also suggested in the multidimensional context by Bourguignon and Chakravarty (2003) and satisfied by the intertemporal poverty measures suggested by Foster (2009), Bossert, Chakravarty, and D'Ambrosio (2012) and Gradin, Del Rio, and Canto (2011). Under *strong focus* p is not sensitive to changes in wellbeing in *any* period above z, for any individual (even if that individual's wellbeing lies below z in other periods). We believe that the poverty analyst may or may not wish to invoke this stronger property; we do not motivate or invoke it in the present section but return to it in section 4, where we show that it restricts the poverty analyst's scope to introduce other, possibly desirable, properties.

We are now able to strengthen the monotonicity property. Again, we start by considering constant wellbeing trajectories in C^T. For wellbeings below the critical level z, the poverty analyst should regard as strictly more poor a trajectory of lower wellbeings. The motivation is equivalent to the motivation for strict monotonicity in the cross-sectional poverty measurement literature (Sen, 1976). If the ordering did not have this property then there would be no sensitivity to the depth of poverty with consequent perverse policy implications, in particular an incentive for the policymaker to focus on the alleviation of the poverty of the least poor among the poor.

Property 3 (CW-Strict Monotonicity)**.** *Given* T*,* $\succsim_T$ *satisfies* CW-strict monotonicity *if it satisfies* CW-focus *and for all* $\boldsymbol{x}, \boldsymbol{y} \in C^T$ *such that* $x_1 < y_1 < z$*,* $\boldsymbol{x} \succ_T \boldsymbol{y}$*.*

Intuitively, p must be a strictly decreasing function of the wellbeing level for constant-wellbeing trajectories. All of the intertemporal poverty measures suggested in the recent literature, except the headcount version of the measure suggested by Foster (2009), satisfy this property. However, their monotonicity properties differ markedly away from the space of constant-wellbeing trajectories. We *may* wish to extend the strict monotonicity property to the entire space of poor trajectories;

Property 4 (Strict Monotonicity)**.** *Given* T*,* $\succsim_T$ *satisfies* STRICT MONOTONICITY *if, for any* $\boldsymbol{x}, \boldsymbol{y} \in \Phi^T$ *such that* $x_t \leq y_t$ *for all* $t \in \{1, 2, \ldots, T\}$ *and* $x_\tau < y_\tau$ *for some* $\tau \in \{1, 2, \ldots, T\}$*,* $\boldsymbol{x} \succ_T \boldsymbol{y}$*.*

Or, just to the space of below-critical-wellbeing in every period trajectories:

Property 5 (Below-z Strict Monotonicity)**.** *Given* T*,* $\succsim_T$ *satisfies* STRICT MONOTONICITY *if, for any* $\boldsymbol{x}, \boldsymbol{y} \in \Phi^T$ *such that* $x_t \leq y_t < z$ *for all* $t \in \{1, 2, \ldots, T\}$ *and* $x_\tau < y_\tau$ *for some* $\tau \in \{1, 2, \ldots, T\}$*,* $\boldsymbol{x} \succ_T \boldsymbol{y}$*.*

Note that there is a certain amount of ambiguity in the current literature. For example, Foster (2009) and Mendola, Busetta, and Milito (2011) specify a *time monotonicity* property, whereby increasing the number of periods 'in poverty' (in our framework with $x_t < z$) increases the poverty measure. This property is in fact entailed by *strict monotonicity* and by *below-z strict monotonicity* and does not need to be specified as a separate property or axiom.

We have introduced the above focus and monotonicity principles in weak forms that are uncontroversial in the context of intertemporal poverty measurement and satisfied by all of the intertemporal poverty measures suggested in the recent literature. We now introduce a third normative principle, continuity, which is not ubiquitous in the literature and which several recently suggested intertemporal poverty measures do not satisfy.

Our motivation for continuity is that marginal changes in wellbeing should have a marginal effect on the evaluation of poverty. If the trajectory ordering is not continuous then we may always find trajectories which are ordered in a perverse way.[18] For example, consider the trajectories $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

t	1	2	3	4	5
$\boldsymbol{a}$	$z/2$	$z/2$	$z-\epsilon$	$z/2$	$z/2$
$\boldsymbol{b}$	$z/2$	$z/2$	z	$z/2$	$z/2$
$\boldsymbol{c}$	$z-\epsilon$	$z-\epsilon$	$z-\epsilon$	$z-\epsilon$	$z-\epsilon$

for any $\epsilon \in (0, z/2)$. The discontinuous measures suggested by Foster (2009) and Bossert, Chakravarty, and D'Ambrosio (2012) would order these trajectories $\boldsymbol{a} \succ_T \boldsymbol{c} \succ_T \boldsymbol{b}$, whereas the marginal difference between $\boldsymbol{a}$ and $\boldsymbol{b}$ in contrast with the substantial difference between $\boldsymbol{b}$ and $\boldsymbol{c}$ would suggest that a more appropriate ordering, as $\epsilon \to 0$, is $\boldsymbol{a} \succ_T \boldsymbol{b} \succ_T \boldsymbol{c}$. Any continuous, strictly monotone trajectory ordering would yield the latter. Imposing continuity is necessary to ensure that the trajectory ordering is 'well behaved' and does not yield any such perverse orderings.

[18] From a pragmatic perspective we may also note that a discontinuous measure would be excessively sensitive to measurement error, at any point of discontinuity. In the present study we have assumed accurate information and are concerned solely with normative motivations. However, in practical applications, this argument is also important.

Property 6 (Euclidean Continuity)**.** *Given T, $\succsim_T$ satisfies* Euclidean continuity *if, for any trajectory $\boldsymbol{x} \in \mathbb{R}_+^T$ and sequence of trajectories $\boldsymbol{y}_j \in \mathbb{R}_+^T$ with limit $\boldsymbol{y} \in \mathbb{R}_+^T$ such that $\boldsymbol{x} \succsim_T \boldsymbol{y}_j$ for each $j \in \mathbb{N}$, $\boldsymbol{x} \succsim_T \boldsymbol{y}$.*

It follows immediately that any function p representing $\succsim_T$ which satisfies *Euclidean continuity* must be continuous on $\mathbb{R}_+^T$.

Those authors who have suggested poverty measures that have discontinuities at z in each period, Foster (2009),[19] Bossert, Chakravarty, and D'Ambrosio (2012) and Gradin, Del Rio, and Canto (2011),[20] have been seeking to ensure that their measures are sensitive either to the proportion of periods below the critical level z (Foster, 2009) or to the number of contiguous periods below z (Bossert, Chakravarty, and D'Ambrosio, 2012; Gradin, Del Rio, and Canto, 2011). We assert that these properties, which may well be desirable in the context of intertemporal poverty measurement, particularly the measurement of *chronic* poverty, do not inherently conflict with continuity and are not sufficient to abandon the strong argument for continuity. We shall return to explore these properties further in section 4.

We now explore the consequences for the trajectory ordering of the normative properties that we have motivated in this section.

Proposition 2. *Given assumptions 1 – 5, if the trajectory ordering $\succsim_T$ on $\mathbb{R}_+^T$ satisfies the properties* weak monotonicity *(1),* CW-focus *(2),* CW-strict monotonicity *(3) and* Euclidean continuity *(6) then*

1. *there exists a function $p : \mathbb{R}_+^T \to \mathbb{R}_+$ which represents $\succsim_T$ on $\mathbb{R}_+^T$, is bounded above by $p(\mathbf{0})$ and which satisfies $p(\boldsymbol{x}) = 0$ for all non-poor trajectories $\boldsymbol{x} \in \mathbb{R}_+^T \setminus \Phi^T$,*

2. *p is strictly decreasing on C^T, and*

3. *for each poor trajectory $\boldsymbol{x} \in \Phi^T$ there exists a unique $\boldsymbol{y} \in C^T$ such that $\boldsymbol{x} \sim_T \boldsymbol{y}$.*

Proof.

1. Existence of $p : \mathbb{R}_+^T \to \mathbb{R}$ which represents $\succsim_T$ on $\mathbb{R}_+^T$ follows immediately from Proposition 1. As p represents a preorder and is thus defined up to strictly increasing transformations we may without loss of generality take $p(\boldsymbol{z}) = 0$.

   By *weak monotonicity* $\boldsymbol{z} \succsim_T \boldsymbol{y}$ and so $p(\boldsymbol{y}) \leq p(\boldsymbol{z}) = 0$ for all $\boldsymbol{y} \in C^T$ such that $y_1 > z$. Consider a non-poor trajectory $\boldsymbol{x} \in \mathbb{R}_+^T \setminus \Phi^T$. By the definition of 'poor' there exists some $\boldsymbol{y} \in C^T$ such that $\boldsymbol{x} \sim_T \boldsymbol{y}$ and $y_1 > z$. Therefore $p(\boldsymbol{x}) = p(\boldsymbol{y}) \leq 0$.

   Consider a sequence of trajectories $\boldsymbol{y}_i \in C^T$ with limit $\boldsymbol{z}$ such that $y_1 > y_{i1} > z$ for each i. By *CW-focus* $\boldsymbol{y}_i \sim_T \boldsymbol{y}$ for each i, and so $\boldsymbol{y} \succsim_T \boldsymbol{y}_i$. By *Euclidean continuity*, therefore, $\boldsymbol{y} \succsim_T \boldsymbol{z}$ so that $p(\boldsymbol{y}) \geq p(\boldsymbol{z}) = 0$. But we showed above that $p(\boldsymbol{y}) \leq p(\boldsymbol{z})$, therefore $p(\boldsymbol{y}) = p(\boldsymbol{z}) = 0$. But $p(\boldsymbol{x}) = p(\boldsymbol{y})$, therefore $p(\boldsymbol{x}) = 0$.

   By *weak monotonicity* $\mathbf{0} \succsim_T \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathbb{R}_+^T$, so that $p(\mathbf{0}) \geq p(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}_+^T$ and so $p(\mathbf{0})$ is an upper bound for p.

2. This follows immediately from *CW-strict monotonicity.*

3. Consider $\boldsymbol{x} \in \Phi^T$. It follows from the above that $\mathbf{0} \succsim_T \boldsymbol{x} \succsim_T \boldsymbol{z}$. If $\mathbf{0} \sim_T \boldsymbol{x}$ or $\boldsymbol{x} \sim_T \boldsymbol{z}$ we have nothing further to show. Otherwise, $\mathbf{0} \succ_T \boldsymbol{x} \succ_T \boldsymbol{z}$ and in particular $\boldsymbol{x} \succ_T \boldsymbol{z}$. Consider a sequence of trajectories $\boldsymbol{y}_i \in C^T$ such that $\boldsymbol{x} \succ_T \boldsymbol{y}_{i+1} \succ_T \boldsymbol{y}_i \succ_T \boldsymbol{z}$ for each

[19] The discontinuity arises from the 'duration poverty line'.

[20] In both of which the discontinuity arises from the form of the factor which weights according to spell duration.

i. Then by *CW-strict monotonicity* $y_{(i+1)1} < y_{i1} < z$; y_{i1} is a decreasing sequence of real numbers, bounded below by 0, let $\hat{y}$ be its greatest lower bound and let $\hat{\boldsymbol{y}} = (\hat{y}, \hat{y}, \ldots, \hat{y})$. By *Euclidean continuity* $\boldsymbol{x} \succsim_T \hat{\boldsymbol{y}}$. Consider a similar sequence $\boldsymbol{y}_j \in C^T$ such that $\boldsymbol{0} \succ_T \boldsymbol{y}_j \succ_T \boldsymbol{y}_{j+1} \succ_T \boldsymbol{x}$; by $\hat{y}$ is also the least upper bound of the increasing sequence of first elements of $\boldsymbol{y}_j$ and $\hat{\boldsymbol{y}} \succsim_T \boldsymbol{x}$. Therefore $\hat{\boldsymbol{y}} \sim_T \boldsymbol{x}$. □

These normative properties do not fully characterise the trajectory ordering $\succsim_T$ but they do significantly restrict its structure. In section 4 we explore further, stronger properties which may be desirable when measuring poverty over time.

3.2 The Social Ordering

Proposition 2 partially characterises an individual intertemporal poverty measure $p : \mathbb{R}_+^T \to \mathbb{R}_+$, representing a trajectory ordering $\succsim_T$ which satisfies the properties *weak monotonicity* (1), *CW-focus* (2), *CW-strict monotonicity* (3) and *Euclidean continuity* (6). The restriction is strong enough for us now to establish the form of the social ordering.

Many of the properties of the social ordering are inherited directly from the trajectory ordering. Returning to the general form (3) of the intertemporal poverty measure, it remains to establish the strictly increasing functions $g : \mathbb{R} \to \mathbb{R}$ and $G_n : \mathbb{R} \to \mathbb{R}$ (for each $n = 3, 4, \ldots, N$). These represent the poverty analyst's judgement about the distribution of the wellbeing indicator among the poor and her judgement about profiles containing different numbers of individuals, respectively.

From Proposition 2 we see that the poverty analyst considers each trajectory in $\mathbb{R}_+^T$ to be equivalent to some constant-wellbeing trajectory. Having chosen p, she is therefore free only to choose how she orders profiles of constant-wellbeing trajectories. Define a function $y : [0, p(\boldsymbol{0})] \to \mathbb{R}_+$ such that

$$y(p) = \begin{cases} y \text{ such that } p(y, y, \ldots, y) = p \text{ if } p > 0 \\ z \text{ if } p = 0 \end{cases} \tag{4}$$

It follows immediately from Proposition 2 that y is well defined and is continuous and strictly decreasing on $[0, p(\boldsymbol{0})]$. The range of y is $[0, z]$. $y(p(\boldsymbol{x}))$ represents that constant level of wellbeing which the poverty analyst considers to be equivalent to the trajectory $\boldsymbol{x}$. We may now decompose g so that $g = \phi \circ y$ where $\phi : [0, z] \to \mathbb{R}_+$ is any continuous and strictly decreasing function on $[0, z]$ (continuity is necessary to preserve *Euclidean continuity* and strictly decreasing is necessary to preserve *CW-strict monotonicity*).

The poverty analyst should choose ϕ to represent her judgement about distribution of the indicator of wellbeing among the poor. This question has been dealt with in some detail in the literature on cross-sectional poverty measurement.[21] A convex function ϕ represents aversion to inequality among the poor; in applied work, a consensus has formed around the 'poverty-gap-squared' form suggested by Foster, Greer, and Thorbecke (1984); applying that in this context gives $\phi(y) = (1 - y/z)^2$. The intertemporal poverty measures suggested by several authors including Foster (2009) and Gradin, Del Rio, and Canto (2011) invoke the 'poverty-gap-squared' transformation and so coincide with (5) below for profiles of constant-wellbeing trajectories. They do not satisfy the properties we invoke in Proposition 2 and therefore treat non-constant-wellbeing trajectories quite differently.

[21] Note that this is the point at which judgements about the 'marginal benefit' of a unit increase in an individual's level of the cardinal wellbeing indicator should be taken into account.

The assumptions and normative principles introduced thus far do not allow the form of the G_n to be determined. The primary role of the G_n is to represent the poverty analyst's ordering of wellbeing profiles representing different population sizes. Many interesting ethical questions arise from consideration of the value of individuals and population size. These have been analysed by philosophers and economists (see, for example, Blackorby, Bossert, and Donaldson (2005) for a detailed treatment in the context of social choice theory). As these questions are not the focus of the present study we introduce a normative principle of population-size neutrality. (It could be replaced with alternative population principles in further work, to reflect alternative judgements about the value of individual profiles and population size.)

The population-size neutrality principle need only be specified for profiles in which the wellbeing trajectory of each individual is identical, in which context it is very straightforward to interpret and motivate, as the assumptions of population symmetry (4) and variable-population subset consistency (5) introduced in section 2 effectively extend its impact across the whole set of wellbeing profiles $\mathcal{X}_{NT}$. It is analogous to the principle of *proportional additions to persons* (commonly called the population principle) introduced in the context of inequality measurement by Dalton (1920).

Property 7 (Population Size Neutrality)**.** *Given assumptions 1 and 2, the poverty ordering* $\succsim$ *on* $\mathcal{X}_{NT}$ *satisfies the following property:*

> *Given* $T \geq 3$*, for all profiles* $X, Y \in \mathcal{X}_{NT}$ *such that* $\boldsymbol{x}_i = \boldsymbol{x}$ *for some* $\boldsymbol{x} \in \mathbb{R}^T_+$*, for each* $i = 1, 2, \ldots, n(X)$ *and* $\boldsymbol{y}_i = \boldsymbol{x}$ *for each* $i = 1, 2, \ldots, n(Y)$*,* $X \sim Y$*.*

Adding *population size neutrality* to the assumptions of Proposition 2, we observe that the conditions for Foster and Shorrocks (1991) Proposition 1 are satisfied, so that $G_n(\alpha) = \frac{\alpha}{n}$ and our intertemporal poverty measure has the form

$$P : X \mapsto \frac{1}{n(X)} \left(\sum_{i=1}^{n(X)} \phi(y(p(x_{i1}, x_{i2}, \ldots, x_{iT}))) \right). \tag{5}$$

4 Stronger Properties of the Trajectory Ordering: Normative Options and Tradeoffs

This section introduces further properties of the trajectory ordering that are less fundamental in terms of the basic structure of the measure but which may be argued to reflect the particular issues that arise when considering the concept of poverty over time. In particular we consider several tradeoffs that exist between possibly attractive normative properties.

4.1 Intertemporal Transfer

First, we discuss the implicit rate of intertemporal subtitution that is inherent in any measure of poverty that includes at least two time periods (and conforms to the properties in section 2 and 3).

We now consider the sensitivity of the poverty measure to changes in wellbeing in more than one time period. We shall argue that the choices made here are central to the concept of intertemporal poverty measurement and lead to strong restrictions on the form of the function p. We identify desirable but incompatible properties, between which the poverty analyst must make a normative or empirical choice, according to the context of application.

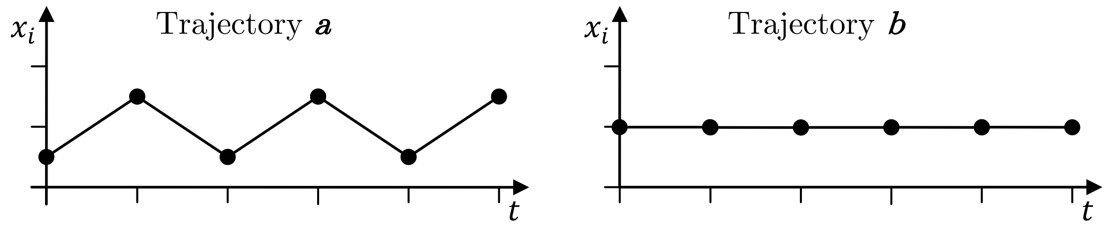

Figure 1: Trajectory A and B: Two trajectories with the same mean; A is more variable than B.

To what extent can a relatively high level of wellbeing in one period compensate for low wellbeing in another? This is a crucial consideration in the choice of a intertemporal poverty measure and one in which there is no consensus as yet in the literature. We note that this is the marginal rate of compensation (or substitution) in any measure, and for two time periods, one could draw 'iso-poverty contours' whose slope would reflect the marginal rate of compensation (or elasticity) between those time periods.[22] For example, trajectory A and trajectory B in figure 1 have the same average wellbeing, but B may very well be preferred to A given that there is fluctuation, and especially if the lower periods mean that the individual falls below the poverty line (in the case of consumption, for example, we would assume if we observed such a trajectory that the individual or household is unable to smooth consumption due to credit constraints).

The intertemporal analogue of Sen's (1976) *transfer* axiom, applied to the trajectory ordering, is as follows.

Property 8 (Intertemporal Transfer). *Given $T \geq 2$, $\precsim_T$ satisfies* INTERTEMPORAL TRANSFER *if, for any $\delta > 0$, $\boldsymbol{x}, \boldsymbol{y} \in \Phi^T$ such that $x_t > x_s$, $y_t = x_t + \delta$, $y_s = x_s - \delta$ for some $t, s \in T$ and $x_\tau = y_\tau$ for all $\tau \neq t, s$, $\boldsymbol{x} \prec_T \boldsymbol{y}$.*

This is similar in spirit to Foster's (2009) *transfer* property which requires chronic poverty to decrease given a smoothing of incomes among those identified as chronically poor. (Foster does not distinguish between smoothing over time and across people.) Note that the measures suggested by Foster (2009) do not in general satisfy the property above the poverty line. Calvo and Dercon's (2009) first, dismissed, suggestion for *'increasing cost of hardship'* is also similar.

Intuitively, *intertemporal transfer* reflects the idea that a period of elevated wellbeing cannot *fully* compensate for a period of depressed wellbeing for an intertemporally poor person, or that the marginal rate of compensation (MRC) implicit in the poverty measure between two periods must be less than perfect. It would seem an appropriate normative choice when the poverty analyst aims to capture the total burden of poverty over time. It is also closely related to the concept of fluctuation or variance aversion, which is natural if the measure is to reflect preferences for smoothing of wellbeing (represented by the diminishing marginal utility of income or consumption).[23] As we noted in the introduction, the measures introduced by Rodgers and Rodgers (1993) and Jalan and Ravallion (2000) do average wellbeing over time and thus allow perfect compensation between periods. In such a case, the isopoverty contours would be represented by straight lines.

[22]This is nicely outlined in the multidimensional context by Bourguignon and Chakravarty (2003) who illustrate with graphics. The analogy to two time periods rather than two dimensions of poverty is straightforward.

[23]Ray (1998) notes that the average utility from a fluctuating income stream is less than that of the utility from the average of that stream.

4.2 Duration or Chronicity of Poverty

We now return to one of the motivating concepts for the measurement of intertemporal poverty; the attempt to capture information about *chronicity* in a quantitative measure. Our main result here is that it is difficult to do this whilst maintaining the transfer properties discussed above. In fact the poverty analyst *must* make a normative choice among these properties according to the context in which she applies the measure and whether she aims to measure the total burden of poverty experienced or chronicity of poverty.

Most of the discussion in the policy literature (CPRC, 2004), and in much of the early economic literature (Calvo and Dercon, 2009) has focused around the concept of chronicity of poverty: prolonged periods below the poverty line must be thought of as worse than shorter, other things equal. For clarity, we noted above that although Foster's (2009) and Mendola, Busetta, and Milito's (2011) *time monotonicity* are motivated in a similar way, they do not quite capture this idea but are in fact directly entailed by *strict monotonicity* and *focus*.

The fundamental (and normatively appealing) idea is that prolonged periods of low consumption or income may have an adverse effect on wellbeing over and above that due to the depth of poverty alone. We attempt to generalise this in a duration-sensitivity property, and show that this measure directly conflicts with the property of *intertemporal transfer* outlined above.

Property 9 (Duration Sensitivity)**.** *Given T and trajectories any $\boldsymbol{x}, \boldsymbol{y}$ with identical average wellbeing, but with strictly more periods in $\boldsymbol{x}$ spent below the poverty line, $p(\boldsymbol{x}) > p(\boldsymbol{y})$.*

Note that, by specifying a duration cutoff τ, whereby if periods in poverty $> \tau$ then the person is considered chronically poor then Foster's (2009) measure has the following implication. A positive transfer δ from a period t where $x_t > z$ to period s where $x_s < z$ could *increase* chronic poverty if $x_t - \delta < z$ and $x_s + \delta < z$ and the number of poor periods in the trajectory is then τ. It is apparent that there is a conflict between *intertemporal transfer* and *duration sensitivity.*

Proposition 3. *No p satisfies both* intertemporal transfer *and* duration sensitivity.

Proof. Consider $\boldsymbol{x}, \boldsymbol{y}$ with identical wellbeings in all periods except $t = 1$ and $t = 2$, where $x_1 = z + \delta$, $y_1 = z - \delta$, $x_2 = z/2 - \delta$ and $y_2 = z/2 + \delta$ where $\delta \in (0, z/4)$. If p satisfies *duration sensitivity* then $p(x) < p(y)$. If p satisfies *intertemporal transfer* then $p(x) \geq p(y)$, a contradiction. Therefore no function p satisfies both *intertemporal transfer* and *duration sensitivity.* □

The tradeoff between *intertemporal transfer* and *duration sensitivity* means that the poverty analyst must choose between them when choosing an intertemporal poverty measure. In fact the only measure proposed thus far in the literature that does satisfy *duration sensitivity* is the duration extended measure proposed by Foster (2009), and only in the circumstance described above. Thus, whilst the concept of chronic poverty in the sense of long duration may be intuitive, in fact imposing this property precludes some other desirable properties.[24] We discuss below another possibility for including a time dimension, that of penalising periods of contiguous poverty.

[24]There is a direct analogy here to the static literature in which the intuitive proposition 'A population with a greater proportion of poor people is worse off than one with a lesser proportion of poor people' conflicts with sensitivity to inequality among the poor, or the transfer principle.

4.3 Intertemporal Compensation

We return now to extend the concept introduced above, of restrictions on the rate of intertemporal compensation or substitution between two time periods. *intertemporal transfer* requires that there should not be perfect compensation when an intertemporal (intrapersonal) transfer is made from a poorer period to a less poor period. We now recognise that the resistance to compensation should in principle be stronger, or at least no weaker, the lower the level of wellbeing experienced.[25] That is, we should allow a greater (or at least, not lesser) marginal rate of compensation (MRC) between a pair of periods when wellbeing is greater in both. The marginal rate of intertemporal compensation between an individual's welfare in two periods should not decrease, as the period wellbeings increase in proportion. Equivalently, the elasticity of compensation should not decrease as wellbeing increases.

Property 10 (Non-Decreasing Compensation)**.** *Consider poor trajectories* $\boldsymbol{x}$ *and* $\alpha\boldsymbol{x}$*,* $\alpha > 1$*. The MRC between any two elements of* $\alpha\boldsymbol{x}$ *should not be higher than those of* $\boldsymbol{x}$*.*

Foster and Santos (2006) introduce a class of measures that possess constant elasticity of substitution (CES) between time periods. These and the measure introduced by Porter and Quinn (2008)[26] are the only measures in the literature that satisfy both properties *non-decreasing compensation* and *intertemporal transfer.* Several papers (Foster, 2009; Calvo and Dercon, 2009; Bossert, Chakravarty, and D'Ambrosio, 2012; Gradin, Del Rio, and Canto, 2011) only allow (imperfect) compensation between periods when both lie below the poverty line. In the cases of Foster (2009) and Calvo and Dercon (2009)[27] the fact that they do not satisfy *non-decreasing compensation* is a consequence of a conflict between this property and their property of *strong focus* mentioned in the previous section. *Strong focus* imposes a great deal of structure on the elasticity of compensation, in fact restricting it to zero relative to any period where wellbeing lies above the poverty line. This implies that above z in any period, the isopoverty contour will be a straight vertical/horizontal line.[28]

Proposition 4. *The trajectory ordering satisfies* strong focus *and* non-decreasing compensation *if and only if it is the 'Rawlsian' ordering.*

Proof. If. Consider $p(\boldsymbol{x})$ that induces the 'Rawlsian' ordering. It clearly satisfies *strong focus* as it is sensitive only to the lowest-period wellbeing. The marginal rate of compensation is zero at all points on the Rawlsian ordering except the constant-wellbeing trajectories at which it is undefined. Consider poor trajectories $\boldsymbol{x}$ and $\alpha\boldsymbol{x}$, $\alpha > 1$, such that all elements of $\boldsymbol{x}$ are strictly less than z while at least one element of $\alpha\boldsymbol{x}$ is strictly greater than z. Elasticity of compensation at all points on $\alpha\boldsymbol{x}$ is zero *and* at $\boldsymbol{x}$ is zero, therefore $p(\boldsymbol{x})$ satisfies *non-decreasing compensation.*

Only if. Suppose there exists $p(\boldsymbol{x})$ that satisfies *strong focus* but is not the Rawlsian ordering. MRC with any element of $\boldsymbol{x}$ that is greater than z must be zero by definition. Consider also poor trajectory $\alpha\boldsymbol{x}$, $\alpha > 1$, such that all elements of $\boldsymbol{x}$ are strictly less than z while at least one element of $\alpha\boldsymbol{x}$ is strictly greater than z, with associated MRC (with any element less than z) zero. By *weak monotonicity*, and as $p(\boldsymbol{x})$ does not represent the Rawlsian ordering, there is some such $\boldsymbol{x}$ with MRC between two elements strictly greater than zero. Comparing $\boldsymbol{x}$ and $\alpha\boldsymbol{x}$, therefore we have increasing compensation and *non-decreasing compensation* cannot hold.

□

[25] We consider this to be intuitively desirable but of course is an empirically testable proposition; we are not aware of a study which has established it.

[26] In an early draft of this current paper.

[27] We refer here to the 'focus first' measures proposed by Calvo and Dercon.

[28] Bourguignon and Chakravarty (2003) illustrate graphically this point for the multidimensional case.

4.4 Permutations of Wellbeings over Time

We briefly discuss here an important issue in the context of intertemporal poverty measurement that is yet to be resolved: the impact on the trajectory ordering of permutations of wellbeings between different time periods.

The simplest approach, which has been taken (implicitly or explicitly) by Foster and Santos (2006), Calvo and Dercon (2009) and Foster (2009), is to impose perfect time symmetry.

Property 11 (Time Symmetry). *Given T, $\precsim_T$ satisfies* time symmetry *if, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^T$ such that $\boldsymbol{x} = M\boldsymbol{y}$ for some permutation matrix M, $\boldsymbol{x} \sim_T \boldsymbol{y}$.*

We may observe that $\precsim_T$ satisfies TIME SYMMETRY if and only if p is a symmetric function of the component wellbeings. Imposing such a property does not allow for various aspects of intertemporal poverty which the poverty analyst might want to capture, including systematic changes (for example a trajectory with a systematic downward trend might be considered 'worse' than an equivalent trajectory with an upward trend) or asymmetric transfer properties (for example the elasticity of compensation may be greater between successive periods than between those separated by a considerable time). This is an especially important consideration when the intervals between data periods are not regular.

Hoy and Zheng (2007), Bossert, Chakravarty, and D'Ambrosio (2012) and Gradin, Del Rio, and Canto (2011) propose versions of an 'intertemporal poverty spell duration sensitivity' property. Gradin, Del Rio, and Canto note: "given any two poverty spells (a certain number of concatenated periods of poverty), the index should be higher when both of the spells are consecutive, ceteris paribus. Thus, the concentration of periods of poverty in a fewer number of poverty spells will increase the individual intertemporal poverty index." Their measure is

$$p(\boldsymbol{x}_i) = \frac{1}{T}\sum_{t=1}^{T}\left(1 - \frac{x_{it}}{z}\right)^{\alpha} w_{it} \tag{6}$$

where

$$w_{it} = \left(\frac{s_{it}}{T}\right)^{\beta} \tag{7}$$

This measure generalises for example Bossert, Chakravarty, and D'Ambrosio (2012). It is clear that due to the weighting (7) an infinitesimal change in x_{it} around the poverty line has a finite impact if period i is contiguous to another period in which wellbeing lies below the poverty line. We simply note here that no measure thus far proposed in the literature satisfies both *continuity* and *spell duration sensitivity* or contiguity, though we have no reason to propose it as an impossibility.

5 Concluding Remarks

The approach to poverty analysis that we have taken in this paper builds firm foundations for the understanding of poverty profiles, and we believe adds considerable depth to the debate on poverty measurement. The analysis provides a more rigorous foundation for the construction of intertemporal poverty measures that enables clarity regarding their properties and thus, we hope, greater clarity in the interpretation and conclusions that may be drawn when they are applied to empirical or policy analysis.

The mild assumptions we invoke in section 2 to establish the trajectory ordering space nevertheless impose a certain structure that is useful for further analysis, in effect developing a very broad class of measures that may be further explored. Although a consensus has been reached in the poverty measurement literature regarding desirable properties of the population-distribution ordering, a similar consensus has not been achieved in the intertemporal context for the properties of the trajectory ordering. In section 3 we proposed what we consider to be some fairly weak normative properties that contribute to the discussion that is beginning in the literature. In section 4, as we discuss above, different (stronger) properties will be necessary in different contexts of application, and these may conflict with each other. The measures suggested in the recent literature certainly do not exhaust the possibilities. We hope that the analytical framework developed in this paper will prove useful for the development of measures with different trajectory ordering properties. In particular, there remains a need for a measure that appropriately captures severe poverty and another that captures chronicity or contiguity of poverty without having to sacrifice continuity.

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