Multidimensional Generalizations of the Relative and Absolute Inequality Indices: The Atkinson–Kolm–Sen Approach*

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This paper generalizes the relative and absolute inequality measures pioneered by Atkinson, Kolm, and Sen to the multidimensional setting with $K$ attributes of well being. Social evaluation functions satisfying a set of axioms which are natural generalizations of their counterparts in the unidimensional context are first introduced. Then it is shown that different scale invariance axioms inexorably lead to classes of relative and absolute inequality indices. Journal of Economic Literature Classification: D31, D63, 131. © 1995 Academic Press, Inc.

1. INTRODUCTION

The recent emphasis on basic needs and human development among economists has put into focus the inadequacy of income as the sole indicator of well being, e.g., Streeten [27] and UNDP [29]. Composite indices of well being have been developed for the purpose of interpersonal or international comparisons, seen e.g., UNDP [29], Maasoumi [21], and Slottje [26]. A logical extension of this area of research seems to be the design of multidimensional inequality measures which summarize the inequalities with respect to different attributes of well being. A small but

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growing literature has emerged, e.g., Maasoumi [20, 21], Hirschberg et al. [14], and Maasoumi and Nickelsburg [22].

In two important papers, Kolm [19] and Atkinson and Bourguignon [5] laid the foundation for the study of multidimensional inequality in the spirit of the social welfare approach pioneered by Atkinson [4], Kolm [16, 17], and Sen [24]. However, they stopped short of suggesting specific functional forms for the measurement of multidimensional inequality. In a pioneering paper, Maasoumi [20] proposed a two-stage approach to the design of a class of multidimensional inequality measures. He first derived an appropriate function $U_i(\cdot)$ to aggregate the welfare attributes for the $i$th person, making use of information theory. Maasoumi suggested that $U_i$ may be interpreted as the "utility" of the $i$th person. In the second stage, a unidimensional inequality index is then chosen to measure the dispersion in $U_i$. In particular, the class of Generalized Entropy (GE) measures was selected,

$$I_i = \frac{1}{N_i(1 + \gamma)} \sum_{i=1}^{N} \left( \frac{NU_i}{\sum U_i} \right)^{1+\gamma} - 1,$$

where $N$ is the number of units and $\gamma$ is a parameter. The purpose of this paper follows the footsteps of Maasoumi and suggests an axiomatic approach to the design of multidimensional inequality measures. Our approach, however, differs from that of Maasoumi by first postulating different sets of axioms and then deriving admissible classes of social evaluation functions and their corresponding inequality indices. The approach in this paper is thus a multidimensional generalization of the Atkinson-Kolm-Sen approach. Interestingly, when the social evaluation function is additively separable, aggregate functions similar to $U_i$ in Maasoumi's approach may be retrieved from these admissible class of social evaluation

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1 The axiomatic approach introduced here is different from that of Jorgenson and Slesnick [15]. Employing consumer demand theory, the Jorgenson-Slesnick approach first determined an individual's (or household's) welfare measure derived from the basket of goods and services actually consumed. Then inequality measurement is based on the distribution of welfare. Both our method and that of Jorgenson and Slesnick are useful in different circumstances. Jorgenson presupposes that the goods and services consumed have market prices. However, price data are sometimes hard to come by, in which case Jorgenson's approach is not implementable. Furthermore, international comparisons of welfare are often interested in the inequality of such public goods as health and education whose prices are difficult if not impossible to obtain. In these circumstances, the indices derived in this paper are more applicable.

2 See also Blackorby and Donaldson [6, 7].

3 After writing this paper, the author has come across a paper by Professor Dardanoni [10] which enhances our understanding of the approach adopted by Maasoumi [20].
functions. The axioms adopted in this paper are generalizations of those in the unidimensional context. In the final analysis, the question of whether these axioms are desirable involves complex ethical issues and thus cannot be answered on a purely objective basis. The multidimensional nature of our problem further complicates the issue. Thus, it is not the intention of this paper to introduce a set of axioms which is acceptable to all. Rather, the objective of this paper is limited to giving complete characterizations of specific classes of social evaluation functions and the corresponding multidimensional inequality indices so that anyone using these tools will have a clear picture of their ethical implications. Much work remains to be done for other sets of axioms.

To put the results of this paper in perspective, a review of the Atkinson-Kolm-Sen approach seems warranted. In the context of measuring income inequality, the social welfare approach explicitly assumes a social evaluation function for a vector of incomes from which an inequality index is derived. To be exact, let \( W(y) \) be a social evaluation function and \( y = (y_1, \ldots, y_N) \) a vector of incomes of \( N \) individuals. The derivation of the inequality measure \( I(y) \) hinges crucially on the concept of equally distributed equivalent income \( y_e \), which is implicitly defined as

\[
W(y_e, \ldots, y_e) = W(y_1, \ldots, y_N).
\]

Two classes of indices may be defined using \( y_e \). Kolm [16] and later Atkinson [4] introduced the class of relative inequality indices as

\[
I_R(y) = 1 - \frac{y_e}{y_{\mu}}, \quad \mu = \frac{1}{N} \sum y_i.
\]

The reason why the aggregate functions for attributes emerge is largely due to Axiom 5 (see below) which restricts the social evaluation function to be additively separable. Axiom 5 is often implicitly assumed in the unidimensional context in order to derive specific functional forms for the social evaluation function; see, e.g., Blackorby et al. [8]. Additive separability is considered by some as rather restrictive. In the final analysis, whether a separability axiom of some sort is acceptable cannot be resolved by purely logical arguments, as are other axioms in this paper. There is also a priori no reason why Axiom 5 may not be discarded. It follows that one may have sets of axioms without Axiom 5 and yet derive the corresponding classes of social evaluation functions and inequality indices. In this connection, a recent paper [28] of mine has shown that the class of Generalized Entropy (GE) measures may be generalized to the multidimensional context and the implicit social evaluation function is not additively separable. It is my conjecture that replacing Axiom 5 by the decomposability axiom in Shorrocks [25] will lead to a class of non-separable social evaluation functions embedding the class of multidimensional GE measures introduced in Tsui [28]. In other words, an aggregate function of attributes for each person is not always a necessary first step to the design of reasonable multidimensional indices.

The generalization of the Pigou-Dalton principle is a case in point. For an extremely useful discussion of the problems, see Dardanoni [10].
Atkinson assumed that the social evaluation function is homothetic so that \( I_{d}(\lambda y) = I_{d}(y) \), \( \lambda > 0 \). Based on a different notion of inequality, Kolm [16–18] defined the class of absolute inequality indices to be the difference between the mean income and the equally distributed equivalent income:

\[
I_{d}(y) = \mu - y.
\]

(3)

\( I_{d}(y) \) is translation-scale invariant, i.e., \( I_{d}(y + \lambda g) = I_{d}(y) \), \( \varepsilon = (1, 1, \ldots, 1) \). Specific functional forms have been proposed for these two classes of indices when the social evaluation function is additively separable; see, e.g., Atkinson [4] and Kolm [17, 18].

As shown below, the Atkinson–Kolm–Sen approach is susceptible to a multidimensional generalization. Furthermore, if the social evaluation function is additively separable and ratio-scale (translation-scale) invariant (to be explained below), specific functional forms for \( I_{d}(y) \) (\( I_{d}(y) \)) emerge. In Section 2, the properties of multidimensional inequality indices are enumerated. In Section 5, the configuration of the underlying social evaluation function is introduced. In Sections 4 and 5, the multidimensional relative and absolute indices are derived respectively. Section 6 concludes by pointing out future research directions.

2. Properties of Multidimensional Inequality Measures

In this section, the multidimensional inequality index is defined and its properties enumerated. In the unidimensional context, inequality measures often satisfy a set of axioms deemed desirable. Recently, Ameil and Cowell [1] conducted an experiment on university students in different countries to find out the acceptability of a set of commonly invoked axioms as properties of inequality measures. As expected, there is no consensus on the acceptability of these axioms. In the final analysis, inequality involves interpersonal comparison. The axioms are in most cases ethical in nature. In the multidimensional context, there is likely to be even less consensus on the properties of multidimensional indices. We choose those axioms which are generalizations of their counterparts in the unidimensional context and are often considered to have a high degree of acceptability.

In the rest of this section, the multidimensional inequality index is defined. Let the distribution of \( K \) attributes among \( N \) individuals be represented by an \( N \times K \) matrix \( X = [x_{ik}] \), where \( x_{ik} \) is the amount of the \( k \)th attribute possessed by the \( i \)th individual. Thus, the \( i \)th row of \( X \) is the amount of the \( K \) attributes possessed by the \( i \)th individual. A multidimensional inequality index is a real-valued function \( R(\cdot): D \rightarrow R_+ \), where \( D \) is
some subset of the set of $N \times K$ matrices with real elements and $R_+$ is the set of nonnegative real numbers,\textsuperscript{6} and:

(a) $I(\cdot)$ is continuous.

(b) $I(X) = 0$ if $X$ is a matrix with identical rows. When there is complete equality, the degree of inequality is normalized to zero.

(c) $I(X) = I(\Pi X)$, where $\Pi$ is an $N \times N$ permutation matrix; i.e., the "label" of each individual does not count.

(d) $I(BX) < I(X)$, where $B$ is a bistochastic matrix and $BX$ cannot be derived by permuting the rows of $X$.\textsuperscript{7} This is a multidimensional generalization of the Pigou–Dalton principle (see Section 3 for further discussion).

Analogous to the unidimensional case, two classes of multidimensional inequality indices may be distinguished. An index is relative if and only if it satisfies (a) to (d) and:

(e) $I(XC) = I(X)$, where $C = \text{diag}(c_1, \ldots, c_K)$ is a diagonal matrix and $c_k > 0$, $k = 1, 2, \ldots, K$. If the attributes have different ratio scales, rescaling them should not change the degree of inequality.

An index is absolute if and only if it satisfies (a) to (d) and:

(f) $I(X + P) = I(X)$, where $P$ is an $N \times K$ matrix with identical rows; i.e., the $i$th row of $P$ is $(p_1, \ldots, p_k)$ for all $i$. Adding a constant to a given attribute does not change the level of inequality. This is a generalization of Kolm's formalization of the "leftist" notion of equality.

The point of departure of the Atkinson–Kolm–Sen approach is to articulate the ethical judgements behind the inequality indices using social evaluation functions. We shall introduce different classes of social evaluation functions which lead to relative and absolute inequality indices.

3. Properties of the Underlying Social Evaluation Function

As a prelude to the derivation of multidimensional inequality indices, different classes of social evaluation functions are specified. A social evaluation function $W(\cdot): D \rightarrow R$, $D \subseteq M$ where $M$ is the set of $N \times K$ matrices,

\textsuperscript{6} In the case of relative indices, $D$ is assumed to be the set of $N \times K$ matrices with positive elements; otherwise, $D$ is a set of $N \times K$ matrices with real elements.

\textsuperscript{7} A bistochastic matrix $B = [b_{ij}]$, $b_{ij} \geq 0$ is a square matrix such that $\sum_i b_{ij} = 1$ and $\sum_j b_{ij} = 1$. 

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ranks different distributions of the $K$ attributes among the $N$ individuals. The attributes may range from income to such welfare attributes as health or education. It is assumed that $W(X)$ satisfies the following properties:

**Axiom 1.** $W(X)$ is continuous.

**Axiom 2.** $W(X)$ is strictly increasing with the elements of $X$.

**Axiom 3.** $W(\Pi X) = W(X)$, where $\Pi$ is any $N$ by $N$ permutation matrix. This property is often denoted by anonymity (or symmetry) because social welfare is not sensitive to the identity of the individuals.

**Axiom 4.** $W(X)$ is strictly quasi-concave; i.e., $W(\pi X + (1-\pi) Y) > \min\{W(X), W(Y)\}$ for $X \neq Y$, $0 < \pi < 1$.

All these axioms have their counterparts in the unidimensional context; see, e.g., Chakravarty [9]. Axiom 1 restricts the class of social evaluation functions to the class of continuous functions. Axiom 2 is the Pareto principle. Axiom 3 and 4 are included to ensure that $W(\cdot)$ is egalitarian. In this connection, $y$ is defined as more evenly distributed than $x$ if $Y = BX$, where $B$ is a bistochastic matrix and $BX$ cannot be derived by permuting the rows of $X$. To understand the rationale of this definition, it should be noted that, in the unidimensional context, $y = Bx$ if and only if $y$ Lorenz-dominates $x$; equivalently, $y$ may be derived from $x$ by a sequence of Pigou-Dalton transfers, i.e., rank- and mean-preserving transfers from richer to poorer persons without changing the total income (Dasgupta et al. [11]). It is possible to show that $W(BX) > W(X)$ for any $W(\cdot)$ which is anonymous and strictly quasi-concave; see, e.g., Dasgupta et al. [11] and Sen [24].

Returning to Axioms 3 and 4, it is easy to show that, for any anonymous and strictly quasi-concave function such as $W(\cdot)$, $W(BX) > W(X)$ when $BX$ cannot be derived from permuting the rows of $X$. Intuitively, multiplying any $X$ by $B$ renders the distributions of the attributes less spread out.

In the following discussion, the relative and absolute indices will be designed based on a social evaluation function with the above properties. When additive separability and scale invariance axioms are introduced in the next two sections, it is possible to show that unique functional forms of relative and absolute inequality indices emerge.

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*In the case of relative indices, $D = M_{N \times K}$, the set of $N \times K$ matrices with positive elements. This domain restriction is required for the proofs of Theorem 1 below. No such restriction is needed for absolute indices.

*This formulation first appeared in Kolm [19] and is equivalent to his “strongly more equal” criterion; see also Marshall and Olkin [23].

*For the proof of this result, see, e.g., Kolm [19, Theorem 4].
4. **Multidimensional Relative Inequality Indices**

Kolm [19] suggested the multidimensional inequality index

$$I_R(X) = 1 - \delta(X), \quad W(X) = W(\delta(X) X),$$

(4)

where the $i$th row of $X$ is equal to $(\mu_1, ..., \mu_K)$ for all $i$, $\mu_k$, $k = 1, 2, ..., K$, is the mean of the $k$th attribute. The domain of $W(X)$ is restricted to $M_{++}$, the set of matrices with positive real elements. Given the configuration of $W(X)$ as discussed in the previous section, it is not difficult to show that $I_R(X)$ satisfies (a) to (d) in Section 2. The proofs of these properties are simple and are left to the readers. $I_R$ is thus a relative multidimensional inequality index.\(^{12}\)

In the unidimensional context, if the social evaluation function is additively separable, i.e., $W(x)$ is ordinarily equivalent $\Sigma_i U(x_i)$ where $x = (x_1, ..., x_n) \in R^N$, and $U(\cdot)$ is increasing and strictly concave,\(^{13}\) it can be shown that $I_R(x)$ is a relative index if and only if the functional form of $U(\cdot)$ is\(^{14}\)

$$U(x_i) = \begin{cases} 
  a + b(x_i^r/r), & r < 1, \quad r \neq 0 \\
  a + b \log x_i, & r = 0, 
\end{cases}$$

(5)

where $a$ is an arbitrary constant and $b > 0$; the corresponding relative index is

$$I_R(y) = \begin{cases} 
  1 - \left[ 1 - \frac{1}{N} \sum_{i=1}^{N} \left( \frac{x_i}{\mu_i} \right)^{1/r} \right], & r < 1, \quad r \neq 0 \\
  1 - \prod_{i=1}^{N} \left( \frac{x_i}{\mu_i} \right)^{1/N}, & r = 0, 
\end{cases}$$

(6)

\(^{11}\) $\delta(x)$ is in fact formally similar to Debreu's coefficient of resource utilization (Debreu [12]). The properties of Debreu's coefficient of resource utilization are well known; for a recent survey, see, e.g., Ahleim [3].

\(^{12}\) Dardanoni [19] has recently pointed out that when the multidimensional inequality index is defined with respect to the aggregate functions of the individuals as in Maassoumi [20], i.e., $\delta(\cdot) = \{U(s^1), ..., U(s^n)\}$ where $s^i$ is the $i$th row of $X$, multiplying $Y$ by a stochastic matrix $B$ may not result in a fall in $\delta(\cdot)$. Our index does not have this problem because our $\delta(\cdot)$ measures the dispersion of $x'$ and not that of aggregate functions $U_i$; i.e., our index is $\delta(x^1, ..., x^n)$. To show that $\delta(y^1) < \delta(x^1)$ when $W(Y) > W(X)$, $Y = BX$, it is to be noted that $Y_i = X_i$, where elements of each row of $Y_i$ and $X_i$ are the means of the attributes with respect to $Y$ and $X$. Then $W(\delta(X) X) = W(\delta(Y) Y)$, $W(\delta(X) X) = W(\delta(Y) Y)$. Since $W(\cdot)$ is strictly increasing, $\delta(Y) > \delta(X)$. By definition, $I_R(x) = 1 - \delta(X) = I_R(BX)$.

\(^{13}\) $U(\cdot)$ is the same for all individuals, implying that the social evaluation function is anonymous. This is the original formulation of Atkinson [4].

\(^{14}\) See, e.g., Chakravarty [9] for a proof. Chakravarty assumes differentiability of $U(x_i)$. In fact, it is not difficult to show that differentiability may be replaced by continuity.
using the formula in Eq. (2). In the multidimensional setting, a similar result may be proved. In this connection, we assume that there exists a non-singleton set of individuals $S \subset \{1, 2, \ldots, N\}$, such that:

**Axiom 5.** \(W(X) = W(\phi(X^S), X^C)\).

Here \(\phi(\cdot)\) is some continuous function, \(X^S\) is the submatrix of \(X\) including the vectors of attributes of the individuals in \(S\) and \(X^C\) is the complement of \(X^S\). Though additive separability is not explicitly assumed, Axiom 5 and the other axioms are enough to ensure that \(W(\cdot)\) is ordinarily equivalent to \(\sum_i U(x^i)\) where \(U(\cdot): R^K_+ \rightarrow R\) is an increasing and strictly concave function; \(x^i\) is the \(i\)th row of \(X\) consisting of the amounts of attributes possessed by the \(i\)th individual. Next, a ratio-scale invariance axiom is introduced:

**Axiom 6.** \(W(X) = W(Y) \iff W(XC) = W(YC), C = \text{diag}(c_1, \ldots, c_K), c_i > 0, i = 1, 2, \ldots, K\).

An interpretation of Axiom 6 is that the ranking of any two matrices of attributes is preserved if the attributes are rescaled according to their respective ratio scales. This axiom is a generalization of a function being homothetic and is crucial to the derivation of a relative index. This section culminates in the following theorem:

**Theorem 1.** The social evaluation function \(W: M_+ \rightarrow R\), satisfies Axioms 1 to 6 if and only if \(W(X)\) is ordinarily equivalent to \(\sum_i U(x^i)\) where \(U(\cdot): R^K_+ \rightarrow R\) is a strictly increasing concave function assuming the forms

\[
a + b \prod_{k=1}^K x_k^{\alpha_k},
\]

or

\[
a + \sum_{i=1}^K r_k \log x_k,
\]

where \(a\) is an arbitrary constant; the parameters \(b\) and \(r_k\) should be appropriately restricted to ensure that \(U(\cdot)\) is increasing and strictly concave. The corresponding inequality index is relative and has the forms

\[
1 - \left[ \frac{1}{N} \sum_{i=1}^N \prod_{k=1}^K \left( \frac{x_{ik}}{\mu_k} \right)^{\alpha_k} \right]^{1/\Sigma \alpha_k}
\]

\(15\) This formulation of separability in Axiom 5 has its origin in Blackorby et al. [8].
or

\[ 1 - \prod_{i=1}^{N} \left[ \prod_{k=1}^{K} \left( \frac{X_{ik}}{\mu_k} \right)^{\alpha_k Z_i} \right]^{1/N}, \]

(10)

where \( \mu_k \) is the mean value of the \( k \)th attribute.

**Proof.** The proof that \( W(\cdot) \) is additively separable for the case with one attribute may be found in Blackorby et al. [8]. The proof for the case with more than one attribute is identical. It is to be noted that Axiom 5 is strong enough to ensure additively separability because \( W(X) \) satisfies Axiom 3, i.e., anonymity; see Blackorby et al. [8]. To show that \( U(\cdot) \) is strictly concave, let \( B = [1/N] \), i.e., a bistochastic matrix with all its elements equal to \( 1/N \). Since \( W(BX) > W(X) \),

\[ U(\mu_1, ..., \mu_K) > \frac{1}{N} \sum_{i=1}^{N} U(x^i), \]

(11)

where \( x^i \) is the \( i \)th row of \( X \) and \( \mu_k \) is the mean of the \( k \)th attribute. Thus, \( U(\cdot) \) is strictly mid-concave. Continuity of \( U(\cdot) \) implies that \( U(\cdot) \) is strictly concave.

To show that \( U(\cdot) \) must assume the above functional forms as in (7) and (8), assume that there are \( L \) individuals having \( w \) and \( w^* \) and \( M \) individuals having \( \gamma \) and \( \gamma^* \) such that

\[ L \cdot U(w) + M \cdot U(\gamma) = L \cdot U(w^*) + M \cdot U(\gamma^*). \]

(12)

The above equation may be rewritten as

\[ \frac{U(w) - U(w^*)}{U(\gamma) - U(\gamma^*)} = \frac{M}{L}. \]

(13)

Multiplying \( w, w^*, \gamma \) and \( \gamma^* \) by a diagonal matrix \( C = \text{diag}(c_1, ..., c_K) \) and combining with Eq. (13) result in

\[ \frac{U(wC) - U(\gamma C)}{U(w) - U(\gamma)} = \frac{U(w^*C) - U(\gamma^*C)}{U(w^*) - U(\gamma^*)}. \]

(14)

Owing to the continuity of \( U(\cdot) \), there is no loss in generality for the above ratios to be rational numbers. It is thus clear from (14) that

\[ \frac{U(wC) - U(\gamma C)}{U(w) - U(\gamma)} = R(C). \]

(15)

Fixing \( \gamma \), Eq. (15) may be rewritten as the functional equation

\[ U(wC) = R(C) U(w) + P(C). \]

(16)
Since \( U(\cdot) \) is strictly increasing, \( R(C) > 0 \). The solutions of the above functional equation are \( (7) \) or \( (8) \) \((Aczel et al. [2])\). To ensure that \( (7) \) is increasing, \( r_k \) and \( b \) in \( (7) \) must assume the same sign. Furthermore, in order to have \( (7) \) strictly concave, the parameters must be restricted in such a way that

\[
D_1 = U_{11} < 0, \quad D_2 = \begin{vmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{vmatrix} > 0,
\]

\[
(-1)^K D_K = \begin{vmatrix} U_{11} & U_{12} & \cdots & U_{1K} \\ U_{21} & U_{22} & \cdots & U_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ U_{K1} & U_{K2} & \cdots & U_{KK} \end{vmatrix} > 0,
\]

where \( U_y \) is the partial derivative of \( U \) with respect to the \( i \)th and \( j \)th attributes.\(^{16}\) For any \( k \in \{1, 2, \ldots, K\} \),

\[
D_k = \sum \text{sgn} \prod_{j=1}^{k} U_{m_{j,k}},
\]

where \( \{\sigma(1), \ldots, \sigma(K)\} \) is some permutation of \( \{1, 2, \ldots, K\} \) and \( \text{sgn} \) is equal to 1 or \(-1\), depending on whether the permutation is even or odd. To identify the restrictions on \( b \) and \( r_j \), it is useful to note that for any \( k, \ k = 1, 2, \ldots, K \), where

\[
\prod_{j=1}^{k} U_{m_{j,k}}(x_1, \ldots, x_K) = \begin{cases} \left( \prod_{j=1}^{k} b r_{m_{j,k}} \right) \left( \prod_{j=1}^{k} x_{j}^{r_{j}-2} \right) \left( \prod_{j=k+1}^{K} x_{j}^{r_{j}} \right), & k \neq K \\ \left( \prod_{j=1}^{k} b r_{m_{j,k}} \right) \left( \prod_{j=1}^{K} x_{j}^{r_{j}-2} \right), & k = K \end{cases},
\]

\[
r_{m_{j,k}} = \begin{cases} r_{m_{j,k}}, & j \neq \sigma(j) \\ r_{j}(r_{j}-1), & j = \sigma(j). \end{cases}
\]

Then the parameter restrictions may be determined using the formula for \( D_k \). With regard to \( (8) \), the same considerations suggest that, for each \( k, \ k > 0, k = 1, 2, \ldots, K \).

Once the functional forms of \( U(\cdot) \) are determined, it is a trivial exercise to derive the corresponding inequality indices which are obviously relative, i.e., Eqs. \((9)\) and \((10)\).

\(^{16}\) Though differentiability for \( U(\cdot) \) is not assumed, the set of axioms constrains \( W(\cdot) \) to be ordinarily equivalent to the differentiable functional forms in Eqs. \((7)\) and \((8)\).
The proof for the converse is straightforward. Q.E.D.

It is to be noted that the function $U(\cdot)$ above is additively separable (as is Maasoumi’s aggregate function). In a recent paper, Dardanoni has contended that an additive separable $U(\cdot)$ has undesirable properties. However, the functional form for $U(\cdot)$ is a direct result of the set of axioms postulated. It, therefore, seems that the appropriateness of the functional form for $U(\cdot)$ should be judged in terms of the reasonableness of the set of axioms. In fact, if Axiom 5 is replaced by some other axioms, it is obvious that a very different class of social evaluation functions will emerge with different ethical implications. In the final analysis, it is unlikely to have a set of axioms that command a consensus since the problem at hand is inherently ethical. What seems more important is complete characterizations of social evaluation functions and the corresponding inequality indices so that the users of these indices will have a clear picture of the ethical judgements implicit in them.

Unlike unidimensional relative indices, it is not true that a multidimensional inequality measure is relative only if $W(\cdot)$ satisfies Axiom 6. However, if $W(\cdot)$ has the property

$$W(X_\mu) = W(Y_\mu) \Rightarrow W(X_\mu C) = W(Y_\mu C),$$

where $X_\mu$ and $Y_\mu$ are $N \times K$ matrices with completely equal distributions of the $K$ attributes and $C = \text{diag}(c_1, \ldots, c_K)$ is a diagonal matrix, then the “only if” result with respect to relative indices remains valid in the multidimensional setting. Given (17), if $W((1 - I_\rho(X'))X_\mu) = W(X') = W(Y) = W((1 - I_\rho(Y'))Y_\mu)$, where $I_\rho(X')$ is a relative index, $W(XC) = W((1 - I_\rho(X'))X_\mu C) = W((1 - I_\rho(Y'))Y_\mu C) = W(YC)$, implying that $W(\cdot)$ is ratio-scale invariant.

5. MULTIDIMENSIONAL ABSOLUTE INEQUALITY INDEX

In the unidimensional setting, it is possible to show that the social evaluation function is translatable; i.e., there exists a increasing function $\phi(\cdot)$ such that $W(y) = \phi(W_\mu(y))$, where $W_\mu(y + \alpha e) = W_\mu(y) + \alpha$, if and only if $I_\mu(y)$ is an absolute index (see Blackorby and Donaldson [7]). The corresponding functional form for $I_\mu(y)$ is

$$I_\mu(y) = \frac{1}{c} \ln \left[ \frac{1}{N} \sum_{i=1}^{N} \exp[c(\mu_i - y_i)] \right], \quad c > 0. \quad (18)$$

17 See, e.g., Chakravarty [9, p. 41].
The unidimensional absolute index as in (4) is also susceptible to a multidimensional generalization. A multidimensional index \( I_A : M \to R_+ \) is defined as absolute if and only if

\[
I_A(X + P) = I_A(X),
\]

where \( P \) is an \( N \) by \( K \) matrix whose rows are identical. \( I_A \) is defined as

\[
W(X) = W(X_{\mu} - I_A A),
\]

where \( A \) is an \( N \times K \) matrix with all its elements equal to one. \( I_A \) may thus be interpreted as an equal amount of each attribute which society is willing to forego in order to secure a completely equal distribution of the attributes. It is not difficult to show that \( I_A \) satisfies (a) to (d) and (f) in Section 2.

In the case of absolute indices, the following translation scale invariance axiom is crucial:

**Axion 7.** \( W(X) = W(Y) \Leftrightarrow W(X + P) = W(Y + P) \).

Where \( P \) is an \( N \times K \) matrix with identical rows. Adding a constant to each attribute does not change the ranking of social states as prescribed by the social evaluation function. A result analogous to Theorem 1 is summarized in Theorem 2 below.

**Theorem 2.** \( W(X) \) satisfies Axioms 1 to 5 and 7 if and only if \( W(X) \) is ordinally equivalent to \( \sum_i U(x_i) \). \( U(\cdot) \) is a strictly increasing and concave function with the functional form

\[
U(x) = a \prod_{k=1}^{K} \exp(c_k x_k) + b,
\]

where \( a \) is an arbitrary constant; \( b \) and \( c_k \) are parameters whose values are to be chosen such that \( U(\cdot) \) is increasing and strictly concave. The corresponding functional form for \( I_A \) is

\[
\frac{1}{\sum c_k} \ln \left\{ \frac{1}{N} \sum_{i=1}^{N} \exp \left( \sum_{k=1}^{K} c_k (\mu_k - x_{ik}) \right) \right\}.
\]

**Proof:** From Theorem 1, \( W(\cdot) \) is additively separable and \( U(\cdot) \) is increasing and strictly concave. Choose four \( K \) vectors \( y, \xi, y^* \) and \( \xi^* \) such that

\[
L \cdot U(y) + M \cdot U(\xi) = L \cdot U(y^*) + M \cdot U(\xi^*),
\]
where $L$ and $M$ are some integers. Using the same reasoning as in Theorem 1,

$$U(w_1 + p_1, \ldots, w_K + p_K) = R(p) U(w') + Q(p). \quad (24)$$

Let $w'_k \equiv \exp(w_k), p'_k \equiv \exp(p_k), w' \equiv (w'_1, \ldots, w'_K), \text{ and } p' \equiv (p'_1, \ldots, p'_K).$ Then (24) may be rewritten as

$$U'(p'_1 w'_1, \ldots, p'_K w'_K) = R'(p') U'(w') + Q'(p'), \quad (25)$$

where $U'(w'_1 p'_1, \ldots, w'_K p'_K) \equiv U(\ln(w'_1 p'_1), \ldots, \ln(w'_K p'_K)),$ etc. The solution of the above function equation is either

$$U'(w') = a \prod_{i=1}^{K} w_i^{c_i} + b \quad (26)$$

or

$$U'(t') = \sum_{i=1}^{K} c_i \ln t_i' + b \quad (27)$$

(Aczel et al. [2]). Thus, $U(\cdot)$ either has the form as depicted in (21) or is affine. However, the strict concavity of $U(\cdot)$ rules out affine functions. Since $U$ is strictly increasing, $c_k$ and $a$ must have the same sign. Furthermore, $a$ and $c_k$ must be chosen such that $U(\cdot)$ is strictly concave. In this connection,

$$\prod_{i=1}^{K} U_{m,j} = \left( \prod_{i=1}^{K} a_i e_i c_i \left( \prod_{i=1}^{K} \exp(c_j x_i) \right) \right).$$

Using the definition of $D_k$, the parameter restrictions may be derived. The functional form for $I_4$ may then be easily derived. Q.E.D.

6. Conclusion

Theorem 1 and Theorem 2 shows that the absolute and relative inequality indices à la Atkinson, Kolm, and Sen are susceptible to multidimensional generalizations. The configurations of the social evaluation functions behind the indices are analyzed. With the growing trend in the use of multiattribute indices for international or interpersonal comparisons of well being, this paper also suggests exact functional forms of inequality indices for the practitioners.

The domain of the relative indices is $M_{++}$ as opposed to $M$ in the cases of absolute indices so that the relevant functional equations as depicted in
Aczel et al. [2] may be invoked. The restriction helps simplify our results and renders the discussion more translucent. However, in theory, it is easy to extend our results to those cases without domain restriction by using the results in Eichhorn and Gleissner [13].

This paper only investigates the inequality indices corresponding to the additive class of social evaluation functions. This is an assumption frequently adopted in the unidimensional setting. Maasoumi's two-step approach also implicitly assumes a separable social evaluation function by postulating the existence of an aggregate function for each person. His approach has the advantage of highlighting the aggregation issues with respect to the attributes, though the same issues are implicit in the choice of the set of axioms for the social evaluation functions in this paper. Additive separability is a rather restrictive feature. Thus, one future research direction worth exploring is to study the axiomatic foundations of those multidimensional equality indices whose social evaluation function is not additively separable. A clue to the non-separable case is the class of GE measures which is not generated by an additive separable social evaluation function. In a separate paper (Tsui [28]), I have shown that the class of GE measures is susceptible to a multidimensional generalization. It follows that there is an implicit social evaluation function which is not additively separable. It is thus an interesting exercise to find out the corresponding set of axioms. In this connection, a paper by Shorrocks [25] provides some hints.¹⁸ This will be the subject worth exploring in the future.

REFERENCES


¹⁸ I thank Professor Shorrocks for raising this point in a private communication.
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