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INEQUALITY DECOMPOSITION BY FACTOR COMPONENTS[1]

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This paper disaggregates the income of individuals or households into different factor components, such as earnings, investment income, and transfer payments, and considers how to assess the contributions of these sources to total income inequality. In the approach adopted, a number of basic principles of decomposition are proposed and their implications for the assignment of component contributions are examined.

1. INTRODUCTION

JUDGEMENTS ABOUT THE IMPORTANCE of various influences on income inequality have a long history. Over the years it has become increasingly common to relate these judgments to summary indices of inequality, and to attempt to decompose the aggregate inequality value into the relevant component contributions.

The issues to which this kind of analysis has been applied fall into two broad categories. The first category covers those cases where we are interested in the influence of population subgroups, such as those defined by age, sex, or race. Disaggregating inequality by population subgroups raises questions concerning the appropriate decomposition rule and the constraints placed on the choice of inequality measures. These questions have been the subject of a series of recent papers (Bourguignon [**4**], Shorrocks [**12**], Cowell [**5**], Blackorby et al. [**2**], Cowell and Shorrocks [**6**]).

The second category of applications covers situations in which different components of total income are examined. For instance, we may wish to assess the contribution of earnings inequality to that of total income. A number of studies have considered the disaggregation of income into different factor components and proposed methods for decomposing the overall inequality value into the corresponding component contributions (Rao [**11**], Fei et al. [**7**], Fields [**8**], Layard and Zabalza [**9**], Pyatt et al. [**10**]). The decomposition rules used in these studies are essentially ad hoc suggestions that have simple functional representations. This paper examines from a more fundamental viewpoint the problem of identifying the contribution to inequality of any given component of income.

We begin in Section 2 with an introductory discussion of the issues involved in factor decompositions. Section 3 proposes a number of general principles that we might wish to be satisfied by any decomposition rule, and investigates the constraints these place on the contribution assigned to any given factor. The implications of weakening one of the principles (the requirement that the factor contributions sum to the amount of total inequality that needs to be "explained")

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are considered in Section 4. Section 5 summarizes the main results and adds a few concluding remarks.

2. PRELIMINARY DISCUSSION

Some of the underlying issues are readily appreciated if we consider inequality decomposition by factors using the variance as a measure of inequality, a convenient benchmark in this type of problem. Let Y_i^k denote the income of individual $i (i = 1, \ldots, n)$ from source $k (k = 1, \ldots, K)$ and let $\boldsymbol{Y} = (Y_1 \ldots, Y_n) = \sum_k \boldsymbol{Y}^k$ represent the distribution of total incomes. Then

$$(1) \qquad \sigma^2(\boldsymbol{Y}) = \sum_k \sigma^2(\boldsymbol{Y}^k) + \sum_{j \neq k} \sum_k \rho_{jk} \sigma(\boldsymbol{Y}^j) \sigma(\boldsymbol{Y}^k)$$

where ρ_{jk} is the correlation coefficient between $\boldsymbol{Y}^j$ and $\boldsymbol{Y}^k$. If the different types of income are uncorrelated, (1) becomes

$$(2) \qquad \sigma^2(\boldsymbol{Y}) = \sum_k \sigma^2(\boldsymbol{Y}^k)$$

and the obvious decomposition assigns $\sigma^2(\boldsymbol{Y}^k)$ as the contribution of factor k.

If $\rho_{jk} \neq 0$ for some $j \neq k$, we have to deal in some way with the interaction effects between the different factors. One approach would be to introduce separate categories for each of the interaction terms, but this solution raises two further problems: the analysis becomes complicated when even moderate numbers of component factors are distinguished (for example, 10 factors require 55 separate contributions to be specified); and it is not clear how the method of decomposition can be extended to other measures of inequality (apart from those directly related to the variance, such as the coefficient of variation). In this paper we consider decompositions that consist simply of one term corresponding to each source of income. This keeps the number of separate contributions within reasonable limits and allows *all* inequality measures to be decomposed in some fashion. However, as in the variance example above, we will need to determine how the interaction effects should be allocated between the individual factor contributions. As will become apparent, the "natural" decomposition of the variance assigns to factor k half the value of all the interaction terms involving this factor. The contribution of factor k then becomes

$$(3) \qquad S_k^*(\sigma^2) = \sigma^2(\boldsymbol{Y}^k) + \sum_{j \neq k} \rho_{jk} \sigma(\boldsymbol{Y}^j) \sigma(\boldsymbol{Y}^k) = \operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y})$$

and the sum of these contributions over the K types of income gives the aggregate inequality value. We will also find it useful to define $s_k^*(I)$ as the *proportion* of total inequality contributed by factor k when the inequality measure is I. So the proportional contributions in the "natural" decomposition of the variance are given by

$$(4) \qquad s_k^*(\sigma^2) = \frac{S_k^*(\sigma^2)}{\sigma^2(\boldsymbol{Y})} = \frac{\operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y})}{\sigma^2(\boldsymbol{Y})}$$

and these sum to unity.

The variance is rarely used as a measure of inequality, primarily because it is not mean independent (i.e. not invariant to proportional changes in all incomes). However, the same considerations apply to the square of the coefficient of variation, $I_2(Y)$, which is both mean independent and extensively used.[2] If $\rho_{jk} = 0$ for all j, k, we have

$$(5) \qquad I_2(\boldsymbol{Y}) = \frac{\sigma^2(\boldsymbol{Y})}{\mu^2} = \sum_k \frac{\sigma^2(\boldsymbol{Y}^k)}{\mu^2}$$

where μ is the mean of $\boldsymbol{Y}$. So $\sigma^2(\boldsymbol{Y}^k)/\mu^2$ is the obvious choice as the contribution of factor k. When the different types of income are correlated, the interaction terms can be allocated in the same way as for the variance and we obtain the "natural" decomposition of I_2

$$(6) \qquad S_k^*(I_2) = \frac{\operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y})}{\mu^2(\boldsymbol{Y})}.$$

Notice that the proportion of total inequality contributed by factor k is now

$$(7) \qquad s_k^*(I_2) = \frac{S_k^*(I_2)}{I_2(\boldsymbol{Y})} = \frac{\operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y})}{\sigma^2(\boldsymbol{Y})},$$

exactly the same as the proportional contributions derived in equation (4) for the variance. So when we later refer to "the proportional contributions in the natural decomposition of the variance" this is equivalent to (and marginally shorter than) "the proportional contributions in the natural decomposition of the square of the coefficient of variation."

Another useful primary is to outline the decomposition by factor components that has been proposed for the Gini coefficient (see Fei et al. [7, pp. 43–46]). The Gini index can be written

$$(8) \qquad G(\boldsymbol{Y}) = \frac{2}{n^2\mu} \sum_i \left(i - \frac{n+1}{2}\right) Y_i$$

when individuals are indexed by their income rank, so that $Y_1 \leqq Y_2 \leqq \ldots \leqq Y_n$. From (8) we obtain

$$(9) \qquad G(\mathbf{Y}) = \frac{2}{n^2\mu} \sum_i \left(i - \frac{n+1}{2}\right) \sum_k Y_i^k = \sum_k \frac{\mu_k}{\mu} \bar{G}(\boldsymbol{Y}^k)$$

where μ_k is the mean of $\boldsymbol{Y}^k$ and

$$(10) \qquad \bar{G}(\boldsymbol{Y}^k) = \frac{2}{n^2\mu_k} \sum_i \left(i - \frac{n+1}{2}\right) Y_i^k$$

[2]The notation I_2 is used because this index is a member of the Generalized Entropy family I_c given later in equation (19).

is known as the pseudo-Gini for factor k. It is not the conventional Gini value $G(\boldsymbol{Y}^k)$ since the weights attached to Y_i^k correspond to the rank of individual i in the distribution $\boldsymbol{Y}$, which in general is not the same as his rank in the distribution $\boldsymbol{Y}^k$. Equation (9) provides a consistent decomposition of the Gini if the contribution of factor k is defined to be

$$(11)\qquad S_k^*(G) = \frac{\mu_k}{\mu}\bar{G}(\boldsymbol{Y}^k) = \frac{2}{n^2\mu}\sum_i\left(i - \frac{n+1}{2}\right)Y_i^k.$$

Notice that no interaction terms appear in this decomposition. Any such effects have already been implicitly allocated among the component contributions. This suggests that there may be other potential decompositions. It turns out that equation (11) produces the "natural" decomposition rule for the Gini, but alternative methods are available and may be regarded as more attractive.

To investigate the possibility of decomposition rules other than those described above, it is necessary to examine the decomposition problem from a more fundamental viewpoint. This is done in the following section, by specifying general restrictions that might reasonably be imposed on potential decomposition methods.

3. PRINCIPLES OF DECOMPOSITION BY FACTOR COMPONENTS

We begin by assuming that inequality is measured by a function $I(\boldsymbol{Y})$ satisfying the following assumption.

ASSUMPTION 1: (a) $I(\boldsymbol{Y})$ is continuous and symmetric; (b) $I(\boldsymbol{Y}) = 0$ if and only if $\boldsymbol{Y} = \mu\boldsymbol{e}$, where $\boldsymbol{e} = (1, 1, \ldots, 1)$.

If K disjoint and exhaustive components of income are identified, the contribution of factor k to total income inequality can be represented by $S_k(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^K; K)$ where $\boldsymbol{Y}^k$ is the distribution vector for factor k. These contributions are taken to be continuous functions. We also presume that the different factors are treated symmetrically, so no significance is attached to how they are numbered.

ASSUMPTION 2: (a) (Continuity) $S_k(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^K; K)$ is continuous in $\boldsymbol{Y}^k$; (b) (Symmetric Treatment of Factors)

$$S_k(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^K; K) = S_{\pi_k}(\boldsymbol{Y}^{\pi_1}, \ldots, \boldsymbol{Y}^{\pi_K}; K)$$

if $\pi_1, \ldots, \pi_K$ is any permutation of $1, \ldots, K$.

Another reasonable assumption is that the contribution of any one factor should not depend on how many other types of income are distinguished. Otherwise the contribution of earnings might change if capital income was subdivided into rent, interest, and dividends; or if transfer payments were split into pensions, unemployment benefits, and so on. Applying this principle to the first factor implies that we need consider only the 2-way partition of $\boldsymbol{Y}$ into $\boldsymbol{Y}^1$ and $\boldsymbol{Y} - \boldsymbol{Y}^1$. Thus we have the following assumption.

ASSUMPTION 3 (Independence of the Level of Disaggregation):

$$S_1(Y^1, \ldots, Y^K; K) = S_1(Y^1, Y - Y^1; 2) = S(Y^1, Y)$$

and because the factors are treated symmetrically

$$S_k(Y^1, \ldots, Y^K; K) = S(Y^k, Y) \tag{12}$$

where $S(Y^k, Y)$ is continuous by Assumption 2.

For the last of our initial assumptions we require the decomposition to be consistent, in the sense that the contributions sum to the overall amount of inequality.

ASSUMPTION 4 (Consistent Decomposition):

$$\sum_k S_k(Y^1, \ldots, Y^K; K) = \sum_k S(Y^k, Y) = I(Y).$$

At the very least this is a highly desirable property, satisfied in the three cases considered in Section 2 (i.e. the decomposition corresponding to equations (3), (6), and (11)). However it is possible to preserve the essence of all our results when Assumption 4 is replaced by a weaker consistency condition. In order not to introduce unnecessary complications at this stage, discussion of this issue is postponed until Section 4.

THEOREM 1: *Assumptions* 2, 3, *and* 4 *imply*

$$S(Y^k, Y) = a(Y) \cdot Y^k = \sum_i a_i(Y) Y_i^k \tag{13}$$

where

$$I(Y) = a(Y) \cdot Y = \sum_i a_i(Y) Y_i. \tag{14}$$

PROOF: By Assumption 4 we have

$$I(Y) = \sum_{k=1}^{K} S(Y^k, Y) = S(Y^1 + Y^2, Y) + \sum_{k=3}^{K} S(Y^k, Y)$$

if factors 1,2 are first considered separately and then combined into a single factor. Thus

$$S(Y^1 + Y^2, Y) = S(Y^1, Y) + S(Y^2, Y)$$

which, for any given Y, is a Cauchy equation whose solution (Aczel [**1**, Theorem 1, p. 215]) is

$$S(Y^k, Y) = a \cdot Y^k = \sum_i a_i Y_i^k.$$

Since $\boldsymbol{a}$ may depend on the choice of $\boldsymbol{Y}$ we obtain (13) and

$$I(\boldsymbol{Y}) = \sum_k S(\boldsymbol{Y}^k, \boldsymbol{Y}) = \boldsymbol{a} \cdot \sum_k \boldsymbol{Y}^k = \boldsymbol{a} \cdot \boldsymbol{Y}. \qquad Q.E.D.$$

At first sight Theorem 1 appears to provide a remarkably strong result. It suggests that when an inequality index is written as a weighted sum of incomes, the decomposition contribution of factor k is the same weighted sum applied to factor k incomes. Thus a comparison of equations (8) and (14) suggests that for the Gini index

$$(15) \qquad a_i(\boldsymbol{Y}) = 2\left(i - \frac{n+1}{2}\right)/n^2\mu$$

if incomes are ordered so that $Y_1 \leqq Y_2 \leqq \ldots \leqq Y_n$. Therefore

$$(16) \qquad S(\boldsymbol{Y}^k, \boldsymbol{Y}) = \frac{2}{n^2\mu} \sum_i \left(i - \frac{n+1}{2}\right) Y_i^k = \frac{\mu_k}{\mu} \bar{G}(\boldsymbol{Y}^k)$$

where the pseudo-Gini value is defined in (10). This is exactly the factor contribution given by $S_k^*(G)$ in equation (11). Similarly, the variance

$$\sigma^2(\boldsymbol{Y}) = \frac{1}{n} \sum_i (Y_i - \mu) Y_i$$

suggests use of the coefficients

$$(17) \qquad a_i(\boldsymbol{Y}) = \frac{1}{n}(Y_i - \mu)$$

and the contribution of factor k becomes the "pseudo-variance"

$$(18) \qquad S(\boldsymbol{Y}^k, \boldsymbol{Y}) = \frac{1}{n} \sum_i (Y_i - \mu) Y_i^k = \mathrm{cov}(\boldsymbol{Y}^k, \boldsymbol{Y}) = S_k^*(\sigma^2)$$

as obtained in equation (3).

The same procedure can be applied directly to all inequality measures conventionally written in the quasi-separable form of equation (14). This covers all members of the Generalized Entropy family of indices:

$$I_c = \frac{1}{nc(c-1)} \sum_i \left\{ \left(\frac{Y_i}{\mu} \right)^c - 1 \right\}, \qquad c \neq 0, 1,$$

$$(19) \qquad I_1 = \frac{1}{n} \sum_i \frac{Y_i}{\mu} \log \frac{Y_i}{\mu},$$

$$I_0 = \frac{1}{n} \sum_i \log \frac{\mu}{Y_i},$$

which includes the square of the coefficient of variation (I_2) as well as the Theil coefficient (I_1). In the latter case, for example, the form of the index I_1 suggests using the weights

(20) $$a_i(\boldsymbol{Y}) = \frac{1}{n\mu}\log(Y_i/\mu)$$

and the "pseudo-Theil" value gives the factor contributions

(21) $$S(\boldsymbol{Y}^k, \boldsymbol{Y}) = \frac{1}{n\mu}\sum_i \log(Y_i/\mu)Y_i^k = S_k^*(I_1).$$

Equations (16), (18), and (21) may provide the "natural" factor decompositions for the three indices, but they are not the only possibilities, because (14) does not determine the coefficients $a_i(\boldsymbol{Y})$ uniquely. In fact the set of coefficients satisfying (14) form a large set, since the equation imposes just one restriction on the n coefficients. As a consequence the corresponding set of factor decompositions is also large.

To characterize the set of possible decompositions in a convenient way, consider the equations

(22) $$\textstyle\sum_i b_i(\boldsymbol{Y})Y_i = 0,$$

(23) $$\textstyle\sum_i b_i(\boldsymbol{Y}) = 0.$$

For any given $\boldsymbol{Y} \neq \boldsymbol{0}$ equations (22), (23) define a vector space of dimension $n-2$, for which we can choose a basis $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n-2}$. Since this basis will depend on the distribution $\boldsymbol{Y}$, the general solution of (22), (23) is

(24) $$\boldsymbol{b}(\boldsymbol{Y}) = \sum_{j=1}^{n-2} \gamma_j(\boldsymbol{Y})\boldsymbol{\beta}_j(\boldsymbol{Y})$$

where the $\gamma_j(\boldsymbol{Y})$ are arbitrary scalar functions of $\boldsymbol{Y}$ and the $\boldsymbol{\beta}_j(\boldsymbol{Y})$ can be chosen to be continuous in $\boldsymbol{Y}$. Equation (22) alone defines a vector space of dimension $n-1$. By adding the additional base vector $\boldsymbol{\beta}_{n-1}(\boldsymbol{Y})$ to those already constructed, the general solution of (22) can be expressed as

(25) $$\boldsymbol{b}(\boldsymbol{Y}) = \sum_{j=1}^{n-1} \gamma_j(\boldsymbol{Y})\boldsymbol{\beta}_j(\boldsymbol{Y}).$$

Now consider equation (14). For any given $\boldsymbol{Y} \neq \boldsymbol{0}$ this is a single linear restriction on n variables, and the solution can be written

$$\boldsymbol{a}(\boldsymbol{Y}) = \boldsymbol{\alpha}(\boldsymbol{Y}) + \boldsymbol{b}(\boldsymbol{Y}) = \boldsymbol{\alpha}(\boldsymbol{Y}) + \sum_{j=1}^{n-1} \gamma_j(\boldsymbol{Y})\boldsymbol{\beta}_j(\boldsymbol{Y})$$

where $\boldsymbol{\alpha}(\boldsymbol{Y})$ is any particular solution and $\boldsymbol{b}(\boldsymbol{Y})$ satisfies the homogeneous equation (22). Substitution into (13) gives

$$S(\boldsymbol{Y}^k, \boldsymbol{Y}) = \boldsymbol{\alpha}(\boldsymbol{Y})\cdot \boldsymbol{Y}^k + \sum_{j=1}^{n-1} \gamma_j(\boldsymbol{Y})\boldsymbol{\beta}_j(\boldsymbol{Y})\cdot \boldsymbol{Y}^k$$

where continuity of $S(.)$ ensures continuity of $\gamma_j(.)$. Choosing the particular solution

$$(26) \qquad \boldsymbol{\alpha}(\boldsymbol{Y}) = \frac{I(\boldsymbol{Y})}{n\sigma^2(\boldsymbol{Y})}(\boldsymbol{Y} - \mu\boldsymbol{e})$$

and setting $\lambda_j(\boldsymbol{Y}) = \gamma_j(\boldsymbol{Y})/I(\boldsymbol{Y})$, we obtain the following corollary.

COROLLARY 1: *If $\boldsymbol{Y} \neq \mu\boldsymbol{e}$, Assumptions* 1–4 *imply*

$$(27) \qquad S(\boldsymbol{Y}^k, \boldsymbol{Y}) = \left\{ \frac{\operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y})}{\sigma^2(\boldsymbol{Y})} + \boldsymbol{Y}^k \cdot \sum_{j=1}^{n-1} \lambda_j(\boldsymbol{Y})\boldsymbol{\beta}_j(\boldsymbol{Y}) \right\} I(\boldsymbol{Y})$$

where the $\lambda_j(\boldsymbol{Y})$ are arbitrary continuous scalar functions, and $\boldsymbol{\beta}_1(\boldsymbol{Y}), \ldots, \boldsymbol{\beta}_{n-1}(\boldsymbol{Y})$ are linearly independent solutions of (22) *and continuous functions of* $\boldsymbol{Y}$.

Equation (27) implies that there exists a factor decomposition rule corresponding to each selection of the arbitrary functions $\lambda_j(\boldsymbol{Y})$. The set of possible options is therefore large and expands with the number of individuals in the population. A second, more important, implication of (27) concerns the *proportional* contribution of factor k when inequality is measured by $I(.)$. Denoting this by $s_k(I) = S(\boldsymbol{Y}^k, \boldsymbol{Y})/I(\boldsymbol{Y})$, we have from (27)

$$(28) \qquad s_k(I) = s_k^*(\sigma^2) + \boldsymbol{Y}^k \cdot \sum_{j=1}^{n-1} \lambda_j(\boldsymbol{Y})\boldsymbol{\beta}_j(\boldsymbol{Y})$$

where $s_k^*(\sigma^2)$ is given by equation (4). Since the right-hand side of (28) does not depend on the choice of inequality measure, any decomposition proportions obtained with index $I(.)$ can also be obtained using any other index $\hat{I}(.)$, by selecting the same combination of the $\lambda_j(\boldsymbol{Y})$. Thus, for example, setting $\lambda_j(\boldsymbol{Y}) = 0$ for all j, gives

$$s_k(I) = s_k^*(\sigma^2) = s_k^*(I_2)$$

and the decomposition proportions *for any index* correspond to those derived in the natural decomposition of the variance (or I_2). But the situation is perfectly symmetrical, so another selection of $\lambda_j(\boldsymbol{Y})$ would give $s_k^*(G)$, say, on the right-hand side of (28). Then

$$s_k(I) = s_k^*(G) = S_k^*(G)/G(\boldsymbol{Y})$$

and all inequality measures (including the variance and I_2) would generate proportional factor contributions corresponding to those for the natural decomposition of the Gini.

There are two further simple restrictions that can be imposed on factor decompositions.[3]

ASSUMPTION 5: (a) (Population Symmetry) If $\boldsymbol{P}$ is any $n \times n$ permutation matrix, $S(\boldsymbol{Y}^k\boldsymbol{P}, \boldsymbol{Y}\boldsymbol{P}) = S(\boldsymbol{Y}^k, \boldsymbol{Y})$; (b) (Normalization for Equal Factor Distribution) $S(\mu_k \boldsymbol{e}, \boldsymbol{Y}) = 0$ for all μ_k.

Assumption 5(a) is the requirement that the factor contributions do not depend on how individuals in the population are numbered: in other words, we treat individuals symmetrically as in Assumption 1(a). Assumption 5(b) states that the contribution of a factor to total income inequality is zero if all individuals receive the same income from that source. The impact of these two additional assumptions can be seen in terms of the contribution function given in (27).

THEOREM 2: *If* $S(\boldsymbol{Y}^k, \boldsymbol{Y})$ *satisfies equation* (27), *then*: (*i*) *Assumption* 5(*a*) *holds if* $\boldsymbol{\beta}_j(\boldsymbol{YP}) = \boldsymbol{P}^{-1}\boldsymbol{\beta}_j(\boldsymbol{Y})$ *and* $\lambda_j(\boldsymbol{YP}) = \lambda_j(\boldsymbol{Y})$ *for all permutation matrices* $\boldsymbol{P}$. (*ii*) *Assumption* 5(*b*) *holds iff* $\lambda_{n-1}(\boldsymbol{Y}) = 0$ *for all* $\boldsymbol{Y}$.

PROOF: (i) This follows from direct substitution into equation (27).

(ii) By substituting $\boldsymbol{Y}^k = \mu^k \boldsymbol{e}$ into (27), Assumption 4(b) holds iff

$$(29) \qquad 0 = \boldsymbol{e} \sum_{j=1}^{n-1} \lambda_j(\boldsymbol{Y}) \boldsymbol{\beta}_j(\boldsymbol{Y}).$$

But, by construction,

$$\boldsymbol{e}\boldsymbol{\beta}_{n-1}(\boldsymbol{Y}) \neq 0; \qquad \boldsymbol{e}\boldsymbol{\beta}_j(\boldsymbol{Y}) = 0 \qquad (j = 1, \ldots, n-2).$$

So (29) is satisfied iff $\lambda_{n-1}(\boldsymbol{Y}) = 0$. *Q.E.D.*

Theorem 2 indicates that the two extra restrictions contained in Assumption 5 do little to reduce the number of potential decomposition rules. Assumption 5(a) will be satisfied if we ensure that the elements of the basis vectors are permuted in a suitable way when individuals are renumbered.[4] Assumption 5(b) simply reduces by one the degrees of freedom associated with the choice of the functions $\lambda_j(\boldsymbol{Y})$. By substituting into Equation (13), Assumption 5(b) is seen to be equivalent to the requirement that the coefficients $a_i(\boldsymbol{Y})$ sum to zero. This is true for the coefficients obtained in the natural decomposition of both the Gini (equation

[3]We may also wish to insist that the decompositions are "non-trivial," in the sense that the contribution of any particular factor is not independent of how that factor is distributed (given some fixed total income distribution $\boldsymbol{Y}$). It is easily demonstrated that trivial decompositions arise only when $a_i(\boldsymbol{Y})$ is independent of i in equation (13). If this is the case, $a_i(\boldsymbol{Y}) = I(\boldsymbol{Y})/n\mu$ and $s_k(I) = \mu_k/\mu$: in other words the proportional contribution of any income component is equal to the share of that factor in aggregate income.

[4]This condition will be satisfied automatically if the basis vectors are defined in terms of incomes ranked by size, since this ranking is invariant to permutations in the numbering of individuals.

(15)) and the variance (equation (17)), but not for those for the Theil index, I_1 (equation (20)).[5]

If Assumptions 1–5 are accepted, we have a unique decomposition rule for two person populations. The contribution proportions $s_k(I)$ will be invariant to the choice of inequality measure and correspond to those obtained in the natural decomposition of the variance (or Gini, since the expressions are identical with only two individuals). For populations containing three or more people, we have no unique decomposition rule, and it is perhaps worth emphasizing the problems this would generate in any empirical application.

Suppose we have a population with 3 individuals. Take a fixed total income distribution $\boldsymbol{Y}$ and number the individuals so that $Y_1 \leqslant Y_2 \leqslant Y_3$. Now consider the contribution of some fixed distribution $\boldsymbol{Y}^k$ corresponding to a component of total income, using the Gini coefficient as the measure of inequality. For the basis vector $\boldsymbol{\beta}_1$ we may choose $(Y_2 - Y_3, Y_3 - Y_1, Y_1 - Y_2)$. Applying Theorem 2(ii) to equation (27) gives

$$(30) \qquad s_k(G) = \frac{S(\boldsymbol{Y}^k, \boldsymbol{Y})}{G(\boldsymbol{Y})}$$

$$= \frac{\operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y})}{\sigma^2(\boldsymbol{Y})} + \lambda_1(\boldsymbol{Y})\{ Y_1^k(Y_2 - Y_3) + Y_2^k(Y_3 - Y_1) + Y_3^k(Y_1 - Y_2)\}$$

$$= s_k^*(\sigma^2) + \lambda_1(\boldsymbol{Y})\delta$$

where δ is the expression in brackets, assumed to be nonzero. By selecting different values of λ_1 in the interval $(-\infty, \infty)$, we will generate any value of $s_k(G)$ in the interval $(-\infty, \infty)$. Thus the contribution of any factor expressed as a proportion of total inequality can be made to give *any* value between plus and minus infinity!

This is an unsatisfactory state of affairs, to say the least, and one that would inevitably cause havoc in the interpretation of empirical results for factor decompositions. Of course, for a suitable choice of $\lambda_1(\boldsymbol{Y})$ in equation (30) we would obtain the natural decomposition of the Gini. This is the case if $\lambda_1(\boldsymbol{Y}) = \sigma^2(\boldsymbol{Y})(Y_3 - Y_1)/(Y_2 - \mu)$. But there is no compelling reason why this value should be chosen, nor why the "natural" decomposition rules, in general, should

[5] Assumption 5(b) is perhaps questionable, since we may feel that identical positive lump sum transfers are an equalizing force and hence should be associated with a negative contribution to inequality. Accepting this argument would lead us to reject the natural decompositions of the Gini, variance, and I_2. For the Theil index I_1, however, $S(\mu_k \boldsymbol{e}, \boldsymbol{Y}) = -I_0(\boldsymbol{Y})\mu_k/\mu$ where $I_0(.)$ is defined in equation (19). Thus for any $\boldsymbol{Y} \neq \mu\boldsymbol{e}$, $S(\mu_k\boldsymbol{e}, \boldsymbol{Y})$ would give an appropriately negative value.

For the moment we continue to impose Assumption 5(b), but the implication for Theorem 3 of omitting this requirement is provided in footnote 9.

be given special attention.[6] Use of the natural decomposition of the Gini as a means of assessing the relative contributions of income components to total inequality could only be justified by arguing both that the Gini coefficient should be used as the measure of inequality (which is an acceptable position to take), *and* that we must choose the decomposition rule that follows naturally from the conventional way in which the Gini formula is written. This latter proposition is simply untenable.

The multiplicity of potential decomposition rules leads us to search further for reasonable principles to govern the decomposition by factor components. One method of proceeding would be to examine in detail the intuitive interpretations normally attached to statements of the form "factor X contributes Z percent of total inequality." Such an approach is certainly worth pursuing and a few comments are made in the concluding remarks. For the moment, however, we continue by introducing one extra principle, Assumption 6. This assumption is perhaps more contentious than those already introduced. But it turns out to be highly restrictive and sufficient to determine the decomposition rule uniquely.

Suppose the distribution of factor 2 incomes $\boldsymbol{Y}^2$, is simply a permutation of that for factor 1, $\boldsymbol{Y}^1$. If there are other sources of income, we would not necessarily wish to state that factors 1 and 2 make the same contribution to inequality in total incomes. This is because the arrangement of incomes in the vector $\boldsymbol{Y}^1$ may correspond more closely to the distribution of incomes from other sources, $\boldsymbol{Y}-\boldsymbol{Y}^1$ (and hence provides more reinforcement to the inequality arising from factors other than 1), than the arrangement of the same incomes in $\boldsymbol{Y}^2$ does with $\boldsymbol{Y}-\boldsymbol{Y}^2$. This might be expressed intuitively by saying that the correlation between $\boldsymbol{Y}^1$ and $\boldsymbol{Y}-\boldsymbol{Y}^1$ may differ from the correlation between $\boldsymbol{Y}^2$ and $\boldsymbol{Y}-\boldsymbol{Y}^2$. In fact the arrangement of incomes in $\boldsymbol{Y}^2$ may give rise to a negative correlation between $\boldsymbol{Y}^2$ and $\boldsymbol{Y}-\boldsymbol{Y}^2$ (and perhaps a negative decomposition contribution), whilst the correlation between $\boldsymbol{Y}^1$ and $\boldsymbol{Y}-\boldsymbol{Y}^1$ (and the contribution) is positive. However, when there are only two factors these considerations do not apply. The correspondence[7] between $\boldsymbol{Y}^1$ and $\boldsymbol{Y}-\boldsymbol{Y}^1=\boldsymbol{Y}^2$ is exactly the same as the correspondence between $\boldsymbol{Y}^2$ and $\boldsymbol{Y}-\boldsymbol{Y}^2=\boldsymbol{Y}^1$. Thus, there appears no reason why $\boldsymbol{Y}^1$ and $\boldsymbol{Y}^2$ should not be treated symmetrically and assigned the same value in the decomposition.[8]

ASSUMPTION 6 (Two Factor Symmetry): For all permutation matrices $\boldsymbol{P}$,

$$S(\boldsymbol{Y}^1,\boldsymbol{Y}^1+\boldsymbol{Y}^1\boldsymbol{P})=S(\boldsymbol{Y}^1\boldsymbol{P},\boldsymbol{Y}^1+\boldsymbol{Y}^1\boldsymbol{P}).$$

[6] Focusing on "natural" decompositions would, of course, arbitrarily rule out the use of the majority of inequality measures since these are not conventionally written in the form indicated by (14).

[7] The term "correspondence" is intended here to cover any measure of association between two vectors and is not just meant as a synonym for "correlation."

[8] Symmetry of treatment is, of course, explicitly assumed in Assumption 2, although the symmetry requirement here is of a slightly stronger form.

This assumption, combined with those given earlier, provides us with the following result.

THEOREM 3: *Assumptions* 1–6 *imply that*

$$(31)\qquad s_k(I) = \frac{S(Y^k, Y)}{I(Y)} = \frac{\operatorname{cov}(Y^k, Y)}{\sigma^2(Y)} \qquad \text{for all } Y \neq \mu e.$$

PROOF: Let $\Pi_j (j = 1, \ldots, n-2)$ be the set of permutation matrices of the form

$$\Pi_j = \left[\begin{array}{c:c:c} I_{j-1} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \begin{matrix} 010 \\ 001 \\ 100 \end{matrix} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & I_{n-j-2} \end{array}\right]$$

where I_n is the $n \times n$ identity matrix. Then, for all j,

$$(32)\qquad \Pi_j^3 = I_n; \qquad \Pi_j^{\mathrm{T}} = \Pi_j^2.$$

Consider any fixed distribution $Y \neq \mu e$ and suppose $Y^1 = \frac{1}{2} Y(I_n - \Pi_j + \Pi_j^2)$. Then $Y^1 + Y^1\Pi_j = Y$ and $S(Y^1, Y) = S(Y^1\Pi_j, Y)$ by Assumption 6. Substituting equation (27) and applying Theorem 2(ii) we obtain

$$(33)\qquad (Y^1 - Y^1\Pi_j) \cdot \sum_{j=1}^{n-2} \lambda_j \beta_j = 0$$

since $\operatorname{cov}(Y^1, Y) = \operatorname{cov}(Y^1\Pi_j, Y)$. But

$$Y^1 - Y^1\Pi_j = \tfrac{1}{2} Y(I_n - \Pi_j + \Pi_j^2)(I_n - \Pi_j) = Y(\Pi_j^2 - \Pi_j)$$

so (33) can be written

$$(34)\qquad Y(\Pi_j - \Pi_j^2)^{\mathrm{T}} \cdot \sum_{j=1}^{n-2} \lambda_j \beta_j = 0 \qquad (j = 1, \ldots, n-2).$$

For the basis we can choose the linearly independent vectors

$$\beta_j = (\Pi_j - \Pi_j^2) Y^{\mathrm{T}} \qquad (j = 1, \ldots, n-2)$$

since these all satisfy $Y\beta_j = 0 = e\beta_j$ and hence equations (22), (23). Thus (34) becomes

$$\beta_j^{\mathrm{T}} \cdot \sum_{j=1}^{n-2} \lambda_j \beta_j = 0 \qquad (j = 1, \ldots, n-2)$$

which implies

$$\left(\sum_{j=1}^{n-2} \lambda_j \boldsymbol{\beta}_j\right)^{\mathrm{T}} \left(\sum_{j=1}^{n-2} \lambda_j \boldsymbol{\beta}_j\right) = 0.$$

Hence

$$\sum_{j=1}^{n-2} \lambda_j \boldsymbol{\beta}_j = \mathbf{0}$$

and $\lambda_j = 0$ for all j, since $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n-2}$ are linearly independent. Since λ_{n-1} is also zero by Theorem 2(ii), substitution into (27) completes the proof. *Q.E.D.*

Theorem 3 has two important consequences, Firstly, the decomposition rule for any inequality measure is unique. In other words, Assumption 6 has eliminated all the degrees of freedom associated with the choice of the arbitrary functions $\lambda_j(\boldsymbol{Y})$.[9] The second, more interesting, implication is that the *relative* importance of different income components is independent of the choice of inequality measure.[10] We therefore avoid one of the major problems encountered in applied work on distributions: that of having constantly to qualify results by stating that they hold only for the particular index selected. If Assumptions 1–6 are accepted, no such qualifications will be necessary in factor decompositions. The unique and invariant decomposition rule given by (31) corresponds, of course, to the proportions of total inequality attributed to each income source by the natural decomposition of the variance. But this does not mean that the variance is the only measure that can be satisfactorily decomposed: (31) applies equally well to any inequality index.

4. WEAKLY CONSISTENT DECOMPOSITIONS

In this section we return to Assumption 4 and examine the implications of substituting a weaker consistency condition. Instead of requiring that total

[9]The result demonstrated in Theorem 3 is modified slightly if we do not impose Assumption 5. In the $n-1$ dimensional space defined by $\boldsymbol{Y\beta} = 0$, choose $\boldsymbol{\beta}_{n-1}$ to be orthogonal to the $\boldsymbol{\beta}_j$ $(j = 1, \ldots, n-2)$ constructed in the proof. Then $(\boldsymbol{Y}^1 - \boldsymbol{Y}^1\boldsymbol{\Pi}_j)\lambda_{n-1}\boldsymbol{\beta}_{n-1} = 0$ for all $j = 1, \ldots, n-2$, and applying the steps in the proof gives $\lambda_j = 0$ for all $j = 1, \ldots, n-2$. We therefore obtain

$$s_k(I) = \frac{\operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y})}{\sigma^2(\boldsymbol{Y})} + \lambda_{n-1}(\boldsymbol{Y})\boldsymbol{Y}^k\boldsymbol{\beta}_{n-1}.$$

This still leaves the choice of the arbitrary function $\lambda_{n-1}(\boldsymbol{Y})$, so Assumptions 1–4 and 6 are not sufficient to ensure a unique decomposition rule.

[10]This property would not hold if we insisted that only natural decompositions were used, since a change in the inequality measure would simultaneously change the decomposition rule. However the different proportional contributions that would follow from varying the inequality index and decomposition rule simultaneously are *not connected in any way* with the notion that different inequality measures incorporate different "perceptions of inequality," and may therefore lead to a range of results. Any variation obtained in the relative importance attached to factors is caused by a change in the decomposition rule, *not* the choice of index.

inequality be the sum of the factor contributions, which implies

$$S(\mathbf{Y}^1 + \mathbf{Y}^2, \mathbf{Y}) = S(\mathbf{Y}^1, \mathbf{Y}) + S(\mathbf{Y}^2, \mathbf{Y}),$$

we suppose merely that there exists some aggregator function $F(.)$ such that

$$S(\mathbf{Y}^1 + \mathbf{Y}^2, \mathbf{Y}) = F\big(S(\mathbf{Y}^1, \mathbf{Y}), S(\mathbf{Y}^2, \mathbf{Y})\big).$$

More specifically, we replace Assumption 4 with Assumption 4′:

ASSUMPTION 4′: (i) (Weak Consistency)[11] For all $\mathbf{Y}^1, \mathbf{Y}^2, \mathbf{Y}$,

$$S(\mathbf{Y}^1 + \mathbf{Y}^2, \mathbf{Y}) = F\big(S(\mathbf{Y}^1, \mathbf{Y}), S(\mathbf{Y}^2, \mathbf{Y})\big);$$

(ii) $S(\mathbf{0}, \mathbf{Y}) = 0;$

(iii) $S(\mathbf{Y}, \mathbf{Y}) = I(\mathbf{Y}).$

Assumptions 4′(ii) and 4′(iii) are implicit in Assumption 4, but now need to be stated explicitly. Assumption 4′(ii) indicates that there is no contribution to inequality from a (potential) source of income that nobody receives. Assumption 4(iii) states that if there is only one source of income, then all inequality must be attributed to this source.

THEOREM 4: *Assumption 4′ implies that there exists a continuous and strictly monotonic function* $f: \mathbb{R} \to \mathbb{R}$ *such that* $f(0) = 0$ *and*

$$\hat{I}(\mathbf{Y}) = \sum_k \hat{S}(\mathbf{Y}^k, \mathbf{Y}) \tag{35}$$

where

$$\hat{I}(\mathbf{Y}) = f(I(\mathbf{Y})), \tag{36a}$$

$$\hat{S}(\mathbf{Y}^k, \mathbf{Y}) = f\big(S(\mathbf{Y}^k, \mathbf{Y})\big). \tag{36b}$$

PROOF: Let $\mathbf{Y}^1, \mathbf{Y}^2, \mathbf{Y}^3$ be any three factor income distributions. Assumption 4′ implies

$$\begin{aligned} S(\mathbf{Y}^1 + \mathbf{Y}^2 + \mathbf{Y}^3, \mathbf{Y}) &= F\big(S(\mathbf{Y}^1 + \mathbf{Y}^2, \mathbf{Y}), S(\mathbf{Y}^3, \mathbf{Y})\big) \\ &= F\big(F(y^1, y^2), y^3\big) \end{aligned} \tag{37}$$

using the shorthand notation $y^k = S(\mathbf{Y}^k, \mathbf{Y})$. Similarly

$$s(\mathbf{Y}^1 + \mathbf{Y}^2 + \mathbf{Y}^3, \mathbf{Y}) = F\big(y^1, F(y^2, y^3)\big)$$

[11] This assumption and Theorem 4 are similar to the "weak consistency in aggregation" and Lemma 1 of Blackorby and Primont [3, p. 4].

so

$$F\big(F(y^1, y^2), y^3\big) = F\big(y^1, F(y^2, y^3)\big).$$

We know from Assumption 4′(ii) that $F(0, y) = y$. By choosing $Y^2 = -Y^1$, we also know that $F(y^1, y^2) = 0$ has a solution for all y^1. By the Theorem of Aczel [1, p. 254] there exists a continuous and strictly monotonic function $f(.)$ such that

$$(38) \qquad F(y^1, y^2) = f^{-1}(f(y^1) + f(y^2))$$

with $f(0) = 0$, since $F(0, y) = y$. Applying (38) to (37) gives

$$S(Y^1 + Y^2 + Y^3, Y) = f^{-1}\left\{ \sum_{k=1}^{3} f(S(Y^k, Y)) \right\} = f^{-1}\left\{ \sum_{k=1}^{3} \hat{S}(Y^k, Y) \right\}$$

and by further repeated applications of (38) we obtain

$$S\Big(\sum_k Y^k, Y\Big) = f^{-1}\Big\{ \sum_k \hat{S}(Y^k, Y) \Big\}$$

or

$$\hat{I}(Y) = f(I(Y)) = \sum_k \hat{S}(Y^k, Y). \qquad Q.E.D.$$

Theorem 4 demonstrates that any aggregation rule $F(.)$ satisfying Assumption 4 will generate a corresponding function $f(.)$, determined uniquely up to a multiplicative constant, such that the transformed factor contributions $\hat{S}(.)$ sum to the transformed inequality value $\hat{I}(.)$. Conversely different choices of functions $f(.)$ will produce different ways of combining the separate contributions of factors 1 and 2 into the contribution of the two factors taken together. It is not clear how many of these different aggregation rules are likely to be of much use in empirical applications. But we might wish, for example, to consider cases in which the factor contributions combine multiplicatively. Then $F(y^1, y^2) = y^1 y^2$, and we can choose $f(y^1) = \log y^1$. The case examined in Section 3 corresponds, of course, to $F(y^1, y^2) = y^1 + y^2$ and the identity function $f(y^1) = y^1$.

For the purposes of this paper, there is no need to constrain the aggregation rules any further than is done in Assumption 4′. Returning to Section 3 we find that all the assumptions imposed on $I(.)$ and $S(.)$ also hold for $\hat{I}(.)$ and $\hat{S}(.)$ as defined by (36), as long as $\hat{I}(.)$ is substituted for $I(.)$ in Assumption 4. Thus all the results remain intact if we simply replace $S(.)$ with $\hat{S}(.)$ and $I(.)$ with $\hat{I}(.)$.

The discussion of the results in the previous section also applies to the modified factor contributions $\hat{S}(Y, Y)$ except for two minor points. Since weakly consistent decompositions allow transformations of the inequality measure we can obtain "semi-natural" decompositions for inequality measures that are conventionally written as monotonic transformations of the quasi-separable form of equation (14). This applies in particular to the Atkinson family of indices

$$I(Y) = 1 - \left\{ \frac{1}{n} \sum_i \left(\frac{Y_i}{\mu} \right)^{1-\epsilon} \right\}^{1/(1-\epsilon)}, \qquad \epsilon > 0,$$

for which

$$\hat{I}(Y) = \frac{1}{n}\sum_i \left(\frac{Y_i}{\mu}\right)^{1-\epsilon}$$

if we choose $f(y) = (1-y)^{1-\epsilon}$ and the corresponding aggregation rule

$$F(y^1, y^2) = 1 - \left[(1-y^1)^{1-\epsilon} + (1-y^2)^{1-\epsilon}\right]^{1/(1-\epsilon)}.$$

The other small change concerns the implications of Theorem 3. Applying this result, we find that if Y^k, Y are fixed, $\hat{S}(Y^k, Y)/\hat{I}(Y)$ is unique and invariant to the choice of $\hat{I}(.)$. This implies $f(S)/f(I) = \alpha$, say. But the proportional factor contributions S/I will now depend on the aggregation rule chosen, and will not, in general, be invariant to the choice of $I(.)$. For example, combining factor contributions multiplicatively gives rise to $f(x) = \log x$, and hence $f(S)/f(I) = \alpha$ implies $S/I = I^{\alpha-1}$. Of course, if the decomposition is not additive (Assumption 4), the proportional contributions S/I will not, in general, sum to one, although the transformed contributions $\hat{S}/\hat{I}$ do so.

5. CONCLUDING REMARKS

This paper has examined the decomposition of inequality into the contributions associated with different components of income. From the outset it was assumed that the impact of each factor could be summarized in a single term, so the decomposition equation contains K separate contributions when K components of income are identified. Two weak restrictions (Assumptions 2 and 3) ensure that the contribution of factor k can be written as $S(Y^k, Y)$, and therefore depends only on the distribution of factor k incomes and the distribution of total income.

Using the assumption that the contributions sum to the overall inequality value, Theorem 1 demonstrates that $S(Y^k, Y)$ must be a weighted sum of factor k incomes, where the weights are determined by the functional representation of the inequality index used in the decomposition. Applying this result gives "natural" decomposition rules for the variance, Gini coefficient and other indices that are conventionally written in the quasi-separable form. However, alternative decompositions are available because the particular functional representation used for any inequality index is not uniquely determined. These alternatives are equivalent to different choices of the arbitrary functions $\lambda_j(Y)(j-1, \ldots, n-1)$ in equation (27). From the characterization of the options provided by (27) we can state that *any* index can be decomposed in such a way that the proportion of inequality contributed by each source of income is identical to the natural decomposition proportions obtained for any other index.

The additional restrictions made in Assumption 5 do little to resolve the non-uniqueness of potential decomposition methods (unless the population contains just 2 individuals). Assumption 6, however, turns out to be more powerful.

This final restriction requires that two income components be assigned the same contribution to inequality if (a) the distribution of income from both sources is identical and (b) together they make up total income. With this additional constraint, there is a unique decomposition rule for any inequality measure (equation (31)). Furthermore the *proportion* of inequality attributed to each factor is the proportion obtained in the natural decomposition of the variance (or the square of the coefficient of variation). Thus, for each component of income, the assessment of its relative contribution to total income inequality will be *independent of the inequality measure chosen*. This is a particularly attractive feature for those involved in applied research on income distribution.

In Section 4 we replaced the assumption that the factor contributions *sum* to total inequality with the requirement that the contribution of any two factors aggregated together can be derived in some way from their individual contributions. The results obtained earlier remain substantially intact under this weaker aggregation rule.

One issue neglected so far is the relationship between the factor decompositions examined in this paper and the meaning normally attached to statements like "income component k contributes an amount C_k to inequality of total incomes." Two obvious interpretations spring to mind. The contribution C_k of factor k might be regarded as: (A) the inequality which would be observed if income component k was the only source of income differences; (B) the amount by which inequality would fall if differences in factor k income receipts were eliminated. As they stand (A) and (B) are slightly ambiguous. The most satisfactory formal statement of (A) seems to be

$$(39) \qquad C_k^A = I\big(\boldsymbol{Y}^k + (\mu - \mu_k)\boldsymbol{e}\big).$$

In other words, we evaluate a hypothetical distribution in which factor k incomes remain unchanged, but incomes from all other sources are subjected to an egalitarian redistribution. If this hypothetical inequality value is high, or high relative to $I(\boldsymbol{Y})$, then factor k makes a significant contribution. Similarly, a suitable formal statement of (B) is

$$(40) \qquad C_k^B = I(\boldsymbol{Y}) - I\big(\boldsymbol{Y} - \boldsymbol{Y}^k + \mu_k \boldsymbol{e}\big).$$

Here the hypothetical distribution eliminates differences in factor k incomes by replacing the income Y_i^k received by each individual with the average income from that source, μ_k. Factor k then makes a substantial contribution to inequality of total incomes if this equalization of factor k incomes significantly reduces measured inequality. Of course, C_k^B may be negative, in which case factor k tends to compensate for differences in incomes received from other sources.

Consider for the moment using the variance as the measure of inequality. We have

$$C_k^A = \sigma^2\big(\boldsymbol{Y}^k + (\mu - \mu_k)\boldsymbol{e}\big) = \sigma^2(\boldsymbol{Y}^k),$$

$$C_k^B = \sigma^2(\boldsymbol{Y}) - \sigma^2\big(\boldsymbol{Y} - \boldsymbol{Y}^k + \mu_k \boldsymbol{e}\big) = \sigma^2(\boldsymbol{Y}^k) + 2\operatorname{cov}(\boldsymbol{Y}^k, \boldsymbol{Y} - \boldsymbol{Y}^k).$$

Thus interpretation (A) produces the "pure" contribution of factor k and disregards all the potential interaction effects, while (B) allocates all the interaction effects involving factor k to the factor k contribution. In general $C_k^A \neq C_k^B$ and $\sum_k C_k^A \neq \sigma^2(Y) \neq \sum_k C_k^B$, so neither (A) or (B) provide a consistent decomposition rule (except when the Y^k are mutually uncorrelated). On the other hand the contributions derived from equation (31) for the variance do generate a consistent decomposition and the contribution of factor k is simply related to the two intuitive interpretations:

$$S(Y^k, Y) = \text{cov}(Y^k, Y) = \tfrac{1}{2}(C_k^A + C_k^B).$$

Similiarly, using the square of the coefficient of variation, I_2, as the inequality measure gives

$$C_k^A = \frac{\sigma^2(Y^k)}{\mu^2}; \qquad C_k^B = \frac{\sigma^2(Y^k) + 2\,\text{cov}(Y^k, Y - Y^k)}{\mu^2}.$$

The contributions derived from (31) are

$$S(Y^k, Y) = \text{cov}(Y^k, Y) = \tfrac{1}{2}(C_k^A + C_k^B),$$

so the relationship between the contribution rules satisfying (31) and the intuitive interpretations (A), (B) is identical to that obtained with the variance, and equally satisfactory. However for most inequality indices it is difficult to see any obvious connection between $S(Y^k, Y)$ and C_k^A, C_k^B (even if Assumptions 5, 6 are not imposed, so $S(Y^k, Y)$ has only to satisfy equation (27)). This tends to suggest that factor decompositions are only really satisfactory for a limited number of inequality measures.

Another problematic issue concerns the links between different kinds of income. The rationale underlying any factor decomposition requires that we examine each income component separately and neglect the feedback effects on other income sources. Thus, to take the familiar example of tax incidence, we can identify the contribution of taxes to the inequality of post tax incomes, but the computation takes no account of the impact of taxes on the distribution of pre-tax incomes. These indirect effects may, of course, be substantial. However their evaluation requires a specification of behavioral relationships, which are avoided in factor decompositions of the kind examined in this paper. This is both the strength and weakness of this type of analysis.

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