Summer School on Capability and Multidimensional Poverty

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Unidimensional Poverty Measurement

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Main Sources of this Lecture

- Foster and Sen (1997), Annexe of “On Economic Inequality”.
- There are others: please see the readings list.
Sen (1976): Two-Steps

Poverty Measurement involves (at least) two steps. Actually there is also a prior step: where are we going to measure poverty? Here: income/consumption/expenditure.

1. Identification: *Who are the poor?*  
Dichotomises the population: poor/non-poor. Tool: Poverty Line (*z*)

\[ x_i < z \quad \text{Poor} \quad \text{and} \quad x_i \geq z \quad \text{Non-Poor} \]
Types of Poverty Lines

- **Absolute:** Based on the cost of a set of goods and services considered necessary to have a satisfactory life. Usually a food poverty line (2100 calories a day) and a total poverty line are computed. (Cost of Basic Needs Method):
  \[
  \text{Total Poverty Line} = \left(\frac{1}{\text{Engel Coef.}}\right) \times (\text{Basic Food Basket})
  \]

- **Relative:** Defined with reference to the total income of the society. Examples: half of the median income. Problem: Confuses poverty with inequality.

- **Hybrid Lines:** Combinations of absolute and relative lines. Examples:
  - \[ z = (z_r)^\rho(z_a)^{(1-\rho)} \] (Foster, 1998);
  - \[ z = \max(z_a, \alpha + kM) \], with M being the median income, (Ravallion and Chen, 2009).
Sen (1976): Two-Steps

2. **Aggregation**: *How poor are we?*

Construct an index that summarizes the data to form an overall picture of poverty.

- A poverty measure is a function $P: D \rightarrow \mathbb{R}$ which, for each distribution $x$ indicates the level $P(x;z)$ of poverty in the distribution.

- We will adopt an absolute $z$ approach and focus the discussion in terms of the indices.
Difference between inequality and poverty?

• Inequality measurement is intrinsically relative: comparing everyone’s income with those of others.

• Poverty measurement is intrinsically absolute. It is deprivation vis-à-vis- an externally given poverty line z.
Axioms
(Classification of Foster, 2006)

• Invariance Axioms
• Dominance Axioms
• Continuity
• Subgroup Axioms (Consistency and Decomposability)
Invariance Axioms

\( x \) is obtained from \( y \) by a permutation of incomes if \( x = Py \), where \( P \) is a permutation matrix.

Example: \( z = 10 \), \( y = (4, 8, 9, 15) \) \( x = (4, 9, 8, 15) \)

1. **Symmetry (Anonymity):** If \( x \) is obtained from \( y \) by a permutation of incomes, then \( P(x;z) = P(y;z) \).
Invariance Axioms

\( x \) is obtained from \( y \) by a \textit{replication} if the incomes in \( x \) are simply the incomes in \( y \) repeated a finite number of times.

Example: \( z=10, \ y=(4,8,9,15) \)

\( x=(4,4,9,9,8,8,15,15) \)

2. \textit{Replication Invariance (Population Principle)}: If \( x \) is obtained from \( y \) by a \textit{replication}, then \( P(x;z)=P(y;z) \).
Invariance Axioms

\( x \) is obtained from \( y \) by a *an increment to a non-poor person* if:

1. \( x_i > y_i \) for all \( y_i \geq z \)
2. \( x_j = y_j \) for all \( j \neq i \)

Example: \( z=10, y=(4,8,9,15) \) \( x=(4,9,8,16) \)

3. **Focus:** If \( x \) is obtained from \( y \) by *an increment to a non-poor person*, then \( P(x;z)=P(y;z) \).
Invariance Axioms

$(x';z')$ is obtained from $(x;z)$ by a proportional change if $(x';z')=(\alpha x; \alpha z)$ for $\alpha > 0$

Example: $z=10$, $x=(4,8,9,15)$;

$z'=20$, $x'=(8,16,18,30)$

4. \textit{Scale Invariance} (Zero-Degree Homogeneity): If $(x';z')$ is obtained from $(x;z)$ by a proportional change, then $P(x',z')=P(x,z)$. 
Dominance Axioms

$x$ is obtained from $y$ by a *decrement among the poor* if for some $i$:

- $y_i < z$
- we have $x_i < y_i$
- while $x_j = y_j$ for all $j \neq i$

Example: $z=10$, $x=(4,8,9,15)$; $x’=(4,7,9,15)$

4. **Monotonicity:** If $x$ is obtained from $y$ by a decrement among the poor, then $P(x,z) > P(y,z)$. 
Dominance Axioms

Given two distributions $x$ and $y$, with the same mean. We say that $x$ is obtained from $y$ by a **progressive transfer among the poor** if for some $i$ and $j$:

$$y_i < y_j < z$$

we have $$y_i < x_i \leq x_j < y_j$$

while $$x_k = y_k$$ for all $k \neq i, j$

Example: $z=10$, $x=(4,8,9,15)$; $x'=(5,7,9,15)$

5. **Transfer**: If $x$ is obtained from $y$ by a progressive transfer among the poor, then $P(x,z)<P(y,z)$. 
Dominance Axioms

• What if because an increase in the income of a poor, moves him/her above \( z \)? (the number of poor diminishes)

• What if because of a transfer a poor moves him/her above \( z \)? (the number of poor diminishes)
Dominance Axioms

$x$ is obtained from $y$ by an **increment among the poor** if for some $i$:

\[ y_i < z \]

we have \[ x_i > y_i \]

while \[ x_j = y_j \] for all \[ j \neq i \]

Example: \[ z=10, \; x=(4,8,9,15); \; x'=(4,8,11,15) \]

4’. **Strong Monotonicity:** If $x$ is obtained from $y$ by an increment among the poor, then \[ P(x,z) \leq P(y,z). \]
Dominance Axioms

Given two distributions $x$ and $y$, with the same mean. We say that $x$ is obtained from $y$ by a regressive transfer among the poor if for some $i$ and $j$:

$$y_i \leq y_j < z$$

we have $x_i < y_i \leq y_j < x_j$

while $x_k = y_k$ for all $k \neq i, j$

Example: $z=10$, $y=(4,8,9,15)$; $x=(4,6,11,15)$

5’. \textit{Strong Transfer (Sen, 1976)}: If $x$ is obtained from $y$ by a regressive transfer among the poor, then $P(x,z) > P(y,z)$. 
Continuity

**Restricted Continuity**: For any sequence $x^k$ having a fixed non-poor income distribution, if $x^k$ converges to $x$, then $P(x^k;z)$ converges to $P(x;z)$.

**Continuity**: For any sequence $x^k$, if $x^k$ converges to $x$, then $P(x^k;z)$ converges to $P(x;z)$. 
Example of discontinuity in poverty measurement (the Headcount Ratio)

The Indicator Function (base of the Headcount Ratio):

\[ p(x_i; z) = \begin{cases} 
1 & \text{if } x_i < z \\
0 & \text{if } x_i \geq z
\end{cases} \]
Continuity and Censoring

• Focus Axiom – Censored Distribution:
  – It is the distribution in which all incomes above $z$ are replaced by $z$, and the other incomes remain unchanged: $x^* = x^*(z)$
    
    \[
    x^*_i = x_i \quad \text{if} \quad x_i \leq z
    \]
    
    \[
    x^*_i = z \quad \text{if} \quad x_i > z
    \]

• For any continuous measure $P(x;z)$ satisfying the focus axiom: $P(x^*;z) = P(x;z)$

• Censored distributions can also be used to create continuous measures from those satisfying restricted continuity.
Dominance Axioms

- Continuous poverty measures that satisfy invariance axioms satisfy monotonicity and transfer if and only if they satisfy the strong monotonicity and the strong transfer axioms. (Foster, 1984, Donaldson & Weymark, 1986).
Normalisation

If every individual has the poverty line income, then $P(x)=0$
Subgroup Axioms

• **Subgroup Consistency**: If \( P(x';z) > P(x;z) \) and \( P(y';z) = P(y;z) \), and \( n(x') = n(x) \), \( n(y') = n(y) \), then \( P(x',y';z) > P(x,y;z) \).

• Example: \( z=10 \), \( x=(4,8,9,15) \). Suppose two groups: \( x_A=(4,8) \) \( x_B=(9,15) \).

• Say that poverty in A increases: \( x'_A=(3,8) \) while poverty in B remains the same: \( x'_B=(9,15) \).

• Then \( P(x'_A) > P(x_A) \) and \( P(x'_B) = P(x_B) \), and therefore one would expect: \( P(x'_A, x'_B) > P(x_A, x_B) \), ie: \( P(3,8,9,15) > P(4,8,9,15) \).
Subgroup Consistency

• For which practical reason may it be important?
• Evaluation of poverty reduction programs!
• It can be seen as an extension of monotonicity:
  – Monotonicity requires poverty to fall when one person’s poverty level is reduced. SC requires aggregate poverty to fall when one group’s poverty level is reduced.
Subgroup Axioms

• **Additive Subgroup Decomposability**: A poverty measure $P$ is decomposable if:

$$P(x, y) = \frac{n(x)}{n} P(x) + \frac{n(y)}{n} P(y)$$

(Extensible to any number of groups)

Then, one can calculate the contribution of each group to overall poverty:

$$C_x = \left( \frac{n(x)}{n} \right) P(x)/P(x, y)$$

Decomposability implies consistency.
The converse does not hold.
Subgroup Consistency & Additive Decomposability

$P$ is a continuous, subgroup consistent poverty index if and only if $P$ is a continuous, increasing transformation of a continuous, decomposable poverty index.

(Foster and Shorrocks, 1991)
Other Axioms

• **Transfer Sensitivity:** If a transfer $t > 0$ of income takes place from a poor person with income $x_i$ to a poor person with income $x_i + d$, then the magnitude of the increase in poverty must be smaller for larger $x_i$. (Kakwani, 1980)
Poverty Measures
TYPES OF POVERTY MEASURES

Partial Measures
- Headcount Ratio
- Income Gap Ratio

Measures based on Poverty Gaps
- Foster Greer Thorbecke’s family of measures (FGT)
- Sen’s Measure

Measures based on Utility Gaps
- Watt’s Measure
- Chakravarty’s family of Measures
- Decomp. Version of Clark-Hemming & Ulph (CHU)’s Measure

Original CHU:
Based on Atkinson’s EDE Income
Partial Poverty Indices:
The Headcount Ratio (% of poor people)

\[ H = \frac{q}{n} \]

- Example: \( x = (4, 8, 9, 15) \); \( z = 10 \)  Then: \( H = \frac{3}{4} \)
  \( x' = (4, 7, 9, 15) \) \( H = \frac{3}{4} \)
  \( x'' = (3, 9, 9, 15) \) \( H = \frac{3}{4} \)

- Insensitive to depth and distribution, ie: violates monotonicity and transfer. Satisfies only restricted continuity.

- Policy implication?
Partial Poverty Indices: Income Gap Ratio
(average shortfall of the poor)

\[ I = \frac{z - \mu_p}{z} \]

- Example: \( x = (4, 8, 9, 15); \quad z = 10; \quad \mu_p = \frac{4 + 8 + 9}{3} = 7; \quad I = \frac{10 - 7}{10} = 0.3 \)
- Sensitive to depth, i.e., satisfies monotonicity.
  \( x' = (4, 7, 9, 15) \quad I = \frac{10 - 6.66}{10} = 0.33 \)
- Insensitive to distribution: violates transfer.
  \( x'' = (3, 9, 9, 15) \quad I = \frac{10 - 7}{10} = 0.3 \)
- Satisfies only restricted continuity.
- But has counterintuitive policy implications: the number of poor must not decrease!
  \( x''' = (4, 8, 11, 13) \quad I = \frac{10 - 6}{10} = 0.4 \)
Poverty measures based on poverty gaps

• **Individual Normalised Poverty Gap:**
  Distance to the poverty line in poverty line units
  \[ g_i^* = \frac{(z - x_i^*)}{z} \]
  (equivalent: \( g_i = \frac{(z-x_i)}{z} \) if \( x_i < z \) and \( g_i = 0 \) othw.)
  
  Example: \( x = (4, 8, 9, 15); \) \( z = 10 \)
  Gap for individual 1: \( g_1 = \frac{(10-4)}{10} = 0.6 \)
  He falls short 60% of the poverty line.
Poverty Indices: Sen’s Measure (1976)

\[ S(x; z) = \frac{1}{n} \sum_{i=1}^{n} \frac{(2r_i - 1)}{q} g_i^* \]

- \( r_i \) is the ranking of person \( i \) among the poor. The poorest person receives a \( r_i = q \), the next poorest person receives a rank of \( r_i = q - 1 \), and so on until the ‘richest’ poor receives \( r_i = 1 \).
- It satisfies invariance and dominance axioms in the weak versions. Satisfies restricted continuity.
- It violates decomposability and subgroup consistency.
Poverty Indices:
Sen’s Measure (1976)

- Example: \( x = (4, 8, 9, 15); \) \( z = 10. \) Then \( q = 3 \)

\[
S(x; z) = \frac{1}{n} \sum_{i=1}^{n} \frac{(2r_i - 1)}{q} g_i^*
\]

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<th>( r_i )</th>
<th>( (2r_i-1)/q )</th>
<th>( g_i )</th>
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Sen’s Measure \( 0.31 \)
Poverty Indices:  
Sen’s Measure (1976)

- For $n$ sufficiently big, Sen’s Measure can be expressed as:

$$S(x, z) = H[I + (1-I)G_p]$$

With $G_p$ being the Gini Coefficient among the poor.
Poverty Indices: Sen’s Measure

\[ S(x; z) = H[I + (1 - I)G_p] \]

- Gini among the poor (Gp)

\[ G(x) = \frac{2(4+5+1)}{(2)(3)^2(21/3)} = \frac{20}{126} = 0.16 \]

- H=3/4

- I=0.3

- \( S=0.75[0.3+(1-0.3)0.16]=0.31 \)
Poverty Indices:
Foster-Greer-Thorbecke 1984 (FGT)

\[
FGT_\alpha(x; z) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{z - x_i^*}{z} \right)^\alpha = \frac{1}{n} \sum_{i=1}^{n} (g_i^*)^\alpha
\]

- Note that it is a family of measures, as parameter \( \alpha \) can take different values.
- Alternative notation:
- Given \( x=(x_1,x_2,\ldots,x_n) \). Define vector of alfa-normalised gaps: \( g^\alpha=(g_1^\alpha,g_2^\alpha,\ldots,g_n^\alpha) \). Then:
- \( FGT_\alpha=\mu(g^\alpha) \) (just the mean of the vector!)
Poverty Indices: FGT

- When $\alpha=0$, $\text{FGT}_0=H$ (Headcount Ratio)
- When $\alpha=1$, $\text{FGT}_1=\text{HI}=P_1$
  (Per Capita Poverty Gap)
- When $\alpha=2$, $\text{FGT}_2=P_2$
  (Per Capita Squared Poverty Gap)
Poverty Indices: FGT

- All FGT members satisfy invariance axioms and restricted continuity.
- When $\alpha > 0$, $\text{FGT}_\alpha$ satisfies continuity and strong monotonicity. Violate transfer.
- When $\alpha > 1$, $\text{FGT}_\alpha$ satisfies continuity, strong monotonicity and strong transfer.
- When $\alpha > 2$, $\text{FGT}_\alpha$ is also transfer sensitive, ie. distribution sensitive.
Example: FGT0 Headcount Ratio

- Example: $x=(4,8,9,15); \ z=10: \ g^0=(1,1,1,0)$
- $P_0= \mu(g^0) = [1+1+1+0]/4 = 0.75$
- Insensitive to depth, ie: violates monotonicity.
  $x’=(4,7,9,15) \ P_0= \mu(g^0) = [1+1+1+0]/4 = 0.75$
- Insensitive to distribution: violates transfer.
  $x’’=(3,9,9,15) \ P_0= \mu(g^0) = [1+1+1+0]/4 = 0.75$
- Violates continuity
Example: FGT1
Per capita Poverty Gap

• Example: $x=(4,8,9,15); z=10$: $g^1=(0.6,0.2,0.1,0)$
• $P_1= \mu(g^1) = \frac{0.6+0.2+0.1+0}{4}=0.225$
  (Also note that $P1=HI$ Indeed: $0.225=0.75*0.3$)
• Sensitive to depth, ie: satisfies monotonicity.
  $x'=(4,7,9,15)$ $P_1=\frac{0.6+0.3+0.1+0}{4}=0.25$
• Insensitive to distribution: violates transfer.
  $x''=(3,9,9,15)$ $P_1=\frac{0.7+0.1+0.1+0}{4}=0.225$
• Satisfies continuity
• Policy Implication?
Example: FGT2
Per capita Squared Poverty Gap

• Example: $x=(4,8,9,15); \ z=10: \ g^2=(0.36,0.04,0.01,0)$
• $P_2=\mu(g^2)=[0.36+0.04+0.01+0]/4=0.1025$
• Sensitive to depth, ie: satisfies monotonicity.
  $x’=(4,7,9,15) \ P_2=\mu(g^2)=[0.36+0.09+0.01+0]/4=0.115$
• Sensitive to distribution: satisfies transfer.
  $x’’=(3,9,9,15) \ P_2=\mu(g^2)=[0.49+0.01+0.01+0]/4=0.127$
• Satisfies continuity
• Policy Implication?
Individual Poverty Functions in a Graphically

The Indicator Function (base of the Headcount Ratio):
\[ p(x_i; z) = 1 \text{ if } x_i < z \]
\[ p(x_i; z) = 0 \text{ if } x_i \geq z \]

The (Individual) Normalized Poverty Gap:
\[ p(x_i; z) = \left(\frac{z - x_i}{z}\right) \]

The (Individual) Squared Normalized Poverty Gap:
\[ p(x_i; z) = \left(\frac{z - x_i}{z}\right)^2 \]
Poverty Indices

• Foster Greer and Thorbecke (1984)-FGT or $P_\alpha$

• For $n$ sufficiently big, $P_2$ can also be expressed as:

$$P_2 = H[I^2 + (1 - I)^2 C_p^2]$$

• With $C_p^2$ being the Squared Coefficient of Variation among the poor.
Poverty Indices: FGT

\[ P_2 = H[I^2 + (1 - I)^2 C_p^2] \]

- Example:
\[ x=(4,8,9,15); \ z=10 \]
\[ CV^2 p = (((1/7)[(7-4)^2+(8-7)^2+(9-7)^2]^{1/2})^2=0.095 \]
\[ FGT_2 = 0.75[(0.3)^2+(1-0.3)^2(0.095)]=0.1025 \]
FGT & Sen’s Measures

• Both based on poverty gaps $g_i$
• Both are sensitive to the number of poor (H), the depth of their poverty (I) and the distribution of income among the poor.

• Key differences:
  ✓ **Continuity** ($P_2$ is cont while $S$ is not)-This can be solved using the censored distribution.
  ✓ **Subgroup consistency and decomposability** ($P_2$ satisfies both while $S$ violates both): This is a consequence of the inequality measure they are related to: Gini vs. $CV^2$. 
Subgroup Consistency and Decomposability – pros and cons

• **Pros:**
  - Allow breaking-down poverty into its constituent components (groups that contribute more).
  - Evaluation of poverty reduction programs: it would be puzzling to have the overall poverty level going up while poverty in each subgroup goes down!

• **Cons:**
  - One may consider that interdependence matters, and that one’s poverty may depend not only on her own shortfall to $z$ but also on her shortfall *vis-à-vis* the shortfall of others (ie: her relative position to others), then subgroups consistency and decomposability may not be such key properties.
Poverty Indices: Clark, Hemming and Ulph (1981)- CHU

\[
C_\beta(x; z) = \begin{cases} 
1 - \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{x_i^*}{z} \right)^\beta \right]^{1/\beta} & \beta \leq 1, \beta \neq 0 \\
1 - \left[ \frac{1}{n} \prod_{i=1}^{n} \left( \frac{x_i^*}{z} \right) \right]^{1/n} & \beta = 0
\end{cases}
\]

- All incomes are normalised by the poverty line
- Takes Atkinson’s EDE income (general mean with \( \beta < 1 \)) among the poor, normalised by the poverty line and compares that to the ‘reference’ 1.
- When \( \beta = 1 \), the measure is just the per capita poverty gap. It is just taking the normalised mean income among the poor and subtracting it from 1.
- When \( \beta < 1 \), it penalises any inequality among the poor (the lower will be the normalised general mean income with respect to the mean).
Poverty Indices: CHU

- Only positive incomes are allowed.
- It satisfies invariance axioms, dominance axioms (for $\beta<1$) and continuity.
- Each member of the family is not decomposable (except for $\beta=1$) but it is subgroup consistent.
- $\beta$ is a measure of ‘aversion to inequality in poverty’. The lower, the higher is the aversion to inequality.
Poverty Indices based on utility gaps


\[
D(x; z) = A(z) \frac{1}{n} \sum_{i=1}^{n} u(z) - u(x^*_i)
\]

• A(z) is a normalisation factor.
Poverty Indices: Watt’s (1969) Index

\[ W(x; z) = \frac{1}{n} \sum_{i=1}^{n} (\ln z - \ln x_i^*) \]

- Can be interpreted as a Dalton type of poverty measure, where \( u(x_i) = \ln(x_i) \), \( A(z) = 1 \).
- Only positive incomes are allowed.
- Satisfies invariance axioms, monotonicity, continuity and decomposability.
Poverty Indices: Chakravarty’s (1983) Index

\[ Ch(x; z) = \frac{1}{n} \sum_{i=1}^{n} \left[ 1 - \left( \frac{x_i^*}{z} \right)^\beta \right] \quad 0 < \beta < 1 \]

- Can be interpreted as a Dalton type of poverty measure, where \( u(x_i) = (x_i)^\beta \) and \( A(z) = 1/z^\beta \)
- Satisfies all invariance, dominance, continuity and decomposability axioms.
Poverty Indices: Decomposable CHU (Atkinson, 1987)

\[
DC_\beta(x; z) = \begin{cases} 
\frac{1}{\beta n} \sum \left[ 1 - \left( \frac{x_i^*}{z} \right)^\beta \right] & \beta \leq 1, \beta \neq 0 \\
\frac{1}{n} \sum (\ln z - \ln x_i^*) & \beta = 0
\end{cases}
\]

- Note that the first line is an extension of Chakravarty’s Index and the second line is Watt’s Index.
- Can be interpreted as a Dalton type of poverty measure, where \( u(x_i) = (x_i)^\beta / \beta \) and \( A(z) = 1/(z^\beta) \) for \( \beta \neq 0 \) and \( u(x_i) = \ln(x_i) \) for \( \beta = 0 \).
- Satisfies all invariance, dominance, continuity and decomposability axioms.