Summary of Useful Formulas of the AF Method

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Achievement Matrix

Cutoff vector & Weights vector

- Where $x_{ij}$ is the achievement of individual $i$ of attribute or dimension $j$.
- $z_j$ is the deprivation cutoff of attribute or dimension $j$.
- $w_j$ is the weight of attribute or dimension $j$ such that:
  $$w_1 + w_2 + \ldots + w_d = d$$

$$X = \begin{bmatrix}
X_{11} & \ldots & X_{1d} \\
X_{21} & \ldots & X_{2d} \\
\vdots & \ddots & \vdots \\
X_{n1} & \ldots & X_{nd}
\end{bmatrix}$$

$$Z = (Z_1, Z_2, \ldots, Z_d)$$

$$W = (w_1, w_2, \ldots, w_d)$$
Deprivation Matrix

Where

- \( g_{ij}^0 = 1 \) if \( x_{ij} < z_j \) (deprived)
- \( g_{ij}^0 = 0 \) if \( x_{ij} \geq z_j \) (non-deprived)

Or equivalently:

\[
g^0_{ij} = \left( \frac{z_j - x_{ij}}{z_j} \right)^0
\]

\[
g^0 = \begin{bmatrix}
g_{11}^0 & \cdots & g_{1d}^0 \\
g_{21}^0 & \cdots & g_{2d}^0 \\
\vdots & \ddots & \vdots \\
g_{n1}^0 & \cdots & g_{nd}^0
\end{bmatrix}
\]

\[
Z = (Z_1, Z_2, \ldots, Z_d)
\]
Raw Dimensional Headcount Ratios

• These are the deprivation rates by dimension, i.e. the proportion of people who are deprived in that dimension.

• It is simply the mean of each column of the deprivation matrix:

\[ H_j = \left( g_{1j}^0 + g_{2j}^0 + \ldots + g_{nj}^0 \right) / n \]
Weighted Deprivation Matrix

Note that we use the same notation as for the deprivation matrix on purpose.

Where

• \( g^0_{ij} = w_j \) if \( x_{ij} < z_j \) (deprived)

• \( g^0_{ij} = 0 \) if \( x_{ij} \geq z_j \) (non-deprived)

• Or equivalently:

\[
g^0_{ij} = w_j \left( \frac{z_j - x_{ij}}{z_j} \right)^0
\]

\[
g^0 = \begin{bmatrix}
g_{11}^0 & \ldots & g_{1d}^0 \\
g_{21}^0 & \ldots & g_{2d}^0 \\
\vdots & \ddots & \vdots \\
g_{n1}^0 & \ldots & g_{nd}^0 \\
\end{bmatrix}
\]

\[
Z = (Z_1, Z_2, \ldots, Z_d)
\]

\[
w = (w_1, w_2, \ldots, w_d)
\]
Deprivation Count Vector

Where the ‘deprivation count’ or score for each person is the sum of her weighted deprivations

- \( c_i = g_{i1} + ... + g_{id} \)

\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{bmatrix}
\]
Identify the poor

Given a poverty cut-off $k$, we compare the deprivation count with the $k$ cutoff and then censor the deprivations of those who were not identified as poor.

$$\rho_k(x_i; z) = 1 \quad \text{if} \quad c_i \geq k \quad \text{poor}$$

$$\rho_k(x_i; z) = 0 \quad \text{if} \quad c_i < k \quad \text{non-poor}$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$
Censored Weighted Deprivation Matrix and Deprivation Count Vector

This is the key matrix (and vector, alternatively) over which we compute the set of AF indicators for $M_0$

$$g^0(k) = \begin{bmatrix} g_{11}^0(k) & \ldots & g_{1d}^0(k) \\ g_{21}^0(k) & \ldots & g_{2d}^0(k) \\ \vdots & \ddots & \vdots \\ g_{n1}^0(k) & \ldots & g_{nd}^0(k) \end{bmatrix} \quad c(k) = \begin{bmatrix} c_1(k) \\ c_2(k) \\ \vdots \\ c_n(k) \end{bmatrix}$$

Where

- $g_{ij}^0(k) = g_0$ (that is $=(w_j)$) if $c_i \geq k$ (deprived & poor)
- $g_{ij}^0(k) = 0$ if $c_i < k$ (deprived or not but non-poor)
- Similarly: $c_i(k) = c_i$ if $c_i \geq k$ and $c_i(k) = 0$ if $c_i < k$
First we focus on the $M_0$ measure and all its related indicators
Headcount Ratio of MD Poverty

It is the proportion of people who have been identified as poor. Thus:

\[ H = \frac{\sum_{i=1}^{n} \rho_k(x_i; z)}{n} = \frac{q}{n} \]

Where q is the number of poor people.

Headcount Ratio is sometimes called the *incidence* of poverty, or the poverty rate.
Integrity (or breadth) of MD Poverty

• It is the average proportion of deprivations in which the poor are deprived.

\[ A = \frac{\sum_{i=1}^{n} c_i(k)}{dq} \]

Note that it is simple to compute:

1) You compute the proportion of total deprivations each poor person has \((c_i(k)/d)\). Note we need to use the censored deprivation count vector, ie: we ignore the deprivations of the non-poor.

2) You take the average of those proportions (that’s why we divide by \(q\), the number of the poor)
Multidimensional Poverty: $M_0$

(Adjusted Headcount Ratio)

• It is the product of incidence and intensity.

\[
M_0 = H \ast A
\]

• Or equivalently, it is the mean of the censored (weighted) deprivation matrix:

\[
M_0 = \mu(g^0(k)) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^0}{nd}
\]
How do we interpret $M_0$?

- $M_0$ is the mean of the (weighted) censored deprivation matrix, i.e.: the sum of all the non-zero entries (each weighted by the corresponding indicator weight) divided by the total number of entries (people x indicators).
How do we interpret $M_0$?

**Weighted Censored Deprivation Matrix:**

1. Its total number of entries provides the total number of deprivations a society can experience (considered indicators $x$ people).
2. Its total number of non-zero weighted entries provides the total number of weighted deprivations that the poor actually experience in that society.
How do we interpret $M_0$?

- $M_0$ Interpretation

  $M_0$ is the ratio of (2)/(1), i.e. the mean of the weighted censored deprivation matrix.

  Thus, it gives the proportion of weighted deprivations that the poor experience in a society of all the total potential deprivations that the society could experience.
How do we decompose $M_0$ by Indicators and Dimensions?

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What is the contribution of deprivation in health to overall poverty? And in nutrition in particular?
How do we decompose $M_0$ by Indicators and Dimensions?

There are two useful but distinct indicators to look at:

1) Censored headcount ratios
2) Contributions by indicators and dimensions.
Censored Headcount Ratios

• How do they differ from ‘raw’ headcount ratios?
• Raw headcount ratios are the % of people who are deprived in a certain indicator.
• Censored headcount ratios are the % of people who are poor and deprived in a certain indicator.
• Careful! Censored headcounts are not the % of the poor deprived in a certain indicator.
Censored Headcount Ratios

• They are simply the mean of each column of the (weighted) censored deprivation matrix divided by the indicator's weight.

\[
H_j^C = \frac{\sum_{i=1}^{n} g_{ij}^0(k)}{w_j n}
\]

• Note that \( M_0 \) is the weighted sum of the censored headcount ratios.

\[
M_0 = \sum_{j=1}^{d} \left( \frac{w_j}{d} \right) H_j^C
\]
Contribution by Indicator and Dimension

• It is the proportion of total poverty which arises from a particular deprivation.

• Recall from the previous slide:

Thus, the contribution of indicator $j$ to overall poverty is given by:

$$M_0 = \sum_{j=1}^{d} \left( \frac{w_j}{d} \right) H_j^C$$

• Thus, the contribution of indicator $j$ to overall poverty is given by:

$$C_j = \frac{(w_j / d) H_j^C}{M_0}$$
Contribution by Indicator and Dimension

• Note: The sum of the contributions of all d indicators needs to add up to 1 (or 100%).

• Whenever the contribution to poverty of a certain indicator widely exceeds its weight, this suggests that there is a relative high deprivation in this indicator in the country. The poor are more deprived in this indicator than in others.

• If there are more than one indicators in a dimension, the dimensional contribution is simply the sum of the indicators’ contribution.
How do we decompose $M_0$ by population subgroups?

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Decomposition by Population Subgroups

If the entire population $X$ (of size $n$) is divided into two subgroups $X_1$ (of size $n_1$) and $X_2$ (of size $n_2$), then overall $M_0$ is the weighted sum of $M_0$ in each subgroup:

$$M_0(X; z) = \left(\frac{n_1}{n}\right)M_0(X_1; z) + \left(\frac{n_2}{n}\right)M_0(X_2; z)$$

Thus, the contribution of subgroup $i$ to overall poverty is

$$C_{Gi} = \frac{(n_i / n)M_0(X_i; z)}{M_0(X; z)}$$
Decomposition by Population Subgroups

• Note that the sum of the contributions of all groups needs to add up to 1 (or 100%).

• Whenever the contribution to poverty of a region or some other group widely exceeds its population share, this suggests that there is a seriously unequal distribution of poverty in the country, with some regions or groups bearing a disproportionate share of poverty.
Generalising the formulas so that they also apply to $M_1$ and $M_2$
$g^\alpha$ Matrix

\[ g^\alpha = \begin{bmatrix}
  g^{\alpha}_{11} & \ldots & g^{\alpha}_{1d} \\
  g^{\alpha}_{21} & \ldots & g^{\alpha}_{2d} \\
  \vdots & \ddots & \vdots \\
  g^{\alpha}_{n1} & \ldots & g^{\alpha}_{nd}
\end{bmatrix} \]

Where

\[ g^{\alpha}_{ij} = w_j \left( \frac{z_j - x_{ij}}{z_j} \right)^\alpha \quad \text{if} \quad x_{ij} < z_j \]

\[ g^{\alpha}_{ij} = 0 \quad \text{if} \quad x_{ij} \geq z_j \]
Censored $g^\alpha$ Matrix
(after identification, done exactly the same as with $M_0$)

$$g^\alpha(k) = \begin{bmatrix}
g_{11}^\alpha(k) & \ldots & g_{1d}^\alpha(k) \\
g_{21}^\alpha(k) & \ldots & g_{2d}^\alpha(k) \\
\vdots & \ddots & \vdots \\
g_{n1}^\alpha(k) & \ldots & g_{nd}^\alpha(k)
\end{bmatrix}$$

Where

$$g_{ij}^\alpha(k) = g_{ij}^\alpha \quad \text{if} \quad c_i \geq k$$

$$g_{ij}^\alpha(k) = 0 \quad \text{if} \quad c_i < k$$
Multidimensional Poverty in general

$M_1$: Adjusted Poverty Gap

$M_2$: Adjusted FGT

\[ M_\alpha = \mu(g^\alpha(k)) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^\alpha}{nd} \]
Note that $M_1$: Adjusted Poverty Gap

\[ M_1 = H \times A \times G \]

Where $G$ is the average poverty gap across all instances in which poor people are deprived

\[
G = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^0}
\]
Note that $M_2$: Adjusted FGT

$$M_2 = H \times A \times S$$

Where $S$ is the average squared poverty gap (average severity of deprivations) across all instances in which poor people are deprived

$$S = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^2}{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^{0}}$$
Contribution by Indicator

• It is the proportion of total poverty which arises from a particular deprivation.

• In general:

\[ M_\alpha = \sum_{j=1}^{d} \frac{\mu(g_{j}^\alpha(k))}{d} \]

• Where:

\[ \mu(g_{j}^\alpha(k)) = \sum_{i=1}^{n} \frac{g_{ij}^\alpha(k)}{n} \]

• Thus, the contribution of indicator j to overall poverty is given by:

\[ C_j = \frac{\mu(g_{j}^\alpha(k))}{dM_\alpha} \]
Contribution by Population Subgroup

- It is the proportion of total poverty which arises from a particular group.

- In general: \[ M_\alpha = \sum_{i=1}^{n} \mu(g_i^{\alpha}(k)) / n \]

- For groups: If the entire population \( X \) (of size \( n \)) is divided into two subgroups \( X_1 \) (of size \( n_1 \)) and \( X_2 \) (of size \( n_2 \)), then overall \( M_0 \) is the weighted sum of \( M_0 \) in each subgroup

\[
M_\alpha(X; z) = \left( \frac{n_1}{n} \right) M_\alpha(X_1; z) + \left( \frac{n_2}{n} \right) M_\alpha(X_2; z)
\]

- Thus, the contribution of subgroup \( i \) to overall poverty is

\[
C_{Gi} = \frac{n_i / n}{M_\alpha(X; z)} M_\alpha(X_i; z)
\]