HDCA Summer School on Capability and Multidimensional Poverty

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Delft University of Technology, Netherlands

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Properties of Multidimensional Poverty Measures

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Oxford Poverty & Human Development Initiative (OPHI)
Focus of This Lecture

Discuss the axiomatic structure considered as ‘desirable’ or ‘convenient’ for the measurement of poverty in the multidimensional context.
Main Sources of this Lecture

• Please see the reading list for others
Preliminaries
Preliminaries

Multiple dimensions

– Standard of living, knowledge, quality of health (referred as ‘achievements’)

**Achievements** of a society or country can be represented by a matrix or joint distribution

Unit of analysis may be individual or household
## Preliminaries

A typical dataset or achievement matrix with 4 dimensions

<table>
<thead>
<tr>
<th></th>
<th>Income</th>
<th>Years of Education</th>
<th>Housing Index</th>
<th>Malnourished</th>
<th>Person</th>
</tr>
</thead>
<tbody>
<tr>
<td>Person 1</td>
<td>700</td>
<td>14</td>
<td>4</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Person 2</td>
<td>300</td>
<td>13</td>
<td>5</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Person 3</td>
<td>400</td>
<td>10</td>
<td>1</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Person 4</td>
<td>800</td>
<td>11</td>
<td>3</td>
<td>No</td>
<td></td>
</tr>
</tbody>
</table>

\[ \mathbf{x} = \begin{bmatrix} 700 \\ 300 \\ 400 \\ 800 \end{bmatrix} \]

\[ \mathbf{z} = \begin{bmatrix} 500 \\ 12 \\ 3 \end{bmatrix} \]

\( \mathbf{z} \) is the vector of poverty lines
Preliminaries

Matrix $x=[x_{ij}]_{n \times d}$ summarizes the joint distribution of $d$ attributes across $n$ individuals.

Row vector $x_i$ denotes the achievements of person $i$ in all $d$ dimensions.

Column vector $x_{.j}$ denotes the achievements of all $n$ persons in dimension $d$.

Vector $z=[z_1,\ldots,z_d]$ be the cut-off vector containing the poverty line of each dimension.
A general achievement matrix

\[ x_{ij} \]: the achievement of individual \( i \) in dimension \( j \)

**Example:**

\[ x_{1d} \]: the achievement of the 1st individual in dimension \( d \)

\[ x_{n1} \]: the achievement of the \( n \)th individual in the first dimension
Multidimensional Poverty Measurement
Measurement

Measurement of multidimensional poverty involves two major steps like unidimensional measurement

- Identification
- Aggregation
First Step: Identification

Identification: Who is multidimensionally poor?

An ‘identification function’, \( \rho \), decides who should be multidimensionally poor

\[
\rho(x_i, z) = \begin{cases} 
1 & \text{if person } i \text{ is multidimensionally poor} \\
0 & \text{if person } i \text{ is not multidimensionally poor}
\end{cases}
\]

Unlike the unidimensional framework, there can be two types of identification procedures

- Counting Approach
- Aggregate Poverty Line Approach
First Step: Identification

Identification: Counting Approach (Two stages)

*First stage*: Determine whether individuals are deprived in each dimension

*Second stage*: Identify if someone is poor based on an identification function (criterion)

*Three types*:
  - Union criterion (if deprived in at least one dimension)
  - Intersection criterion (if deprived in all dimensions)
  - Intermediate criterion
First Step: Identification

*Example*: Constructing first stage ‘Deprivation Matrix’

<table>
<thead>
<tr>
<th></th>
<th>Income</th>
<th>Years of Education</th>
<th>Housing Index</th>
<th>Mal-nourished</th>
<th>Person</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>x</strong></td>
<td>700</td>
<td>14</td>
<td>4</td>
<td>No</td>
<td>Person 1</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>13</td>
<td>5</td>
<td>Yes</td>
<td>Person 2</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>10</td>
<td>1</td>
<td>Yes</td>
<td>Person 3</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>11</td>
<td>3</td>
<td>No</td>
<td>Person 4</td>
</tr>
<tr>
<td><strong>z</strong></td>
<td>500</td>
<td>12</td>
<td>3</td>
<td>No</td>
<td></td>
</tr>
</tbody>
</table>
First Step: Identification

*Example:* Constructing first stage ‘Deprivation Matrix’

Replace entries: 1 if deprived, 0 if not deprived

<table>
<thead>
<tr>
<th>Income</th>
<th>Years of Education</th>
<th>Housing Index</th>
<th>Malnourished</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ g^0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]

\[ z = \begin{bmatrix} 500 \\ 12 \\ 3 \\ \text{No} \end{bmatrix} \]

These entries fall below cutoffs
First Step: Identification

**Identification:** Aggregate Poverty Line Approach

A person is identified as poor if her aggregate achievement falls below an aggregate poverty line.

Let the aggregation function be denoted by $\phi$.

Then,

\[
\rho(x_i,z) = 1 \quad \text{if} \quad \phi(x_i) < \phi(z) \\
\rho(x_i,z) = 0 \quad \text{if} \quad \phi(x_i) \geq \phi(z)
\]

Example consumer expenditure approach

*Note: No deprivation matrix was created in this situation*
Second Step: Aggregation

**Aggregation:** How poor is the society?

Based on the identification criterion, this step construct an index of poverty $P(x;z)$ summarizing the information of the poor (**a censored matrix can be created just as in the unidimensional framework**).
Axioms
Axioms in Multidimensional Context

Two types

1. *Natural extensions* of the unidimensional framework.

2. Axioms specific to the multidimensional context
Natural Extensions

**Symmetry (Anonymity):** If matrix $y$ is obtained from matrix $x$ by a *permutation* of achievements and the poverty lines remain unchanged, then $P(y;z) = P(x;z)$

$y$ is obtained from $x$ by a *permutation* of incomes if $x = Py$, where $P$ is a permutation matrix.

**Example:** $y = Px = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 4 & 4 & 2 \\ 8 & 6 & 3 \end{bmatrix}$
Natural Extensions

**Replication Invariance (Population Principle):** If matrix $y$ is obtained from matrix $x$ by a *replication* and the poverty lines remain unchanged, then $P(y;z) = P(x;z)$

$y$ is obtained from $x$ by a *replication* if each person’s achievement vector in $x$ is simply repeated a finite number of times

*Example:* 

$$x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \quad \quad y = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \\ 8 & 6 & 3 \end{bmatrix}$$
Natural Extensions

Scale Invariance (Homogeneity of Zero-Degree): If all achievements in matrix $x$ and all poverty lines in $z$ are changed by the same proportion $\alpha > 0$, then $P(\alpha x; \alpha z) = P(x; z)$.

**Example:**

$$x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \quad z = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix}$$

$$\alpha x = \begin{bmatrix} 2(4) & 2(4) & 2(2) \\ 2(3) & 2(5) & 2(4) \\ 2(8) & 2(6) & 2(3) \end{bmatrix} \quad \alpha z = \begin{bmatrix} 2(4) & 2(5) & 2(3) \end{bmatrix}$$
Natural Extensions

**Focus**: Unlike in the unidimensional framework, there are two types of focus axiom

*(Type I)* Focus on those identified as multidimensionally poor (we are not interested in those who are not multidimensionally poor)

*(Type II)* Focus on dimensions where multidimensionally poor are deprived (we are not interested in dimensions in which they are not deprived)
Natural Extensions

**Poverty Focus (Type I):** If $y$ is obtained from $x$ by *an increment to a non-poor person’s achievements* and the poverty lines remain unchanged, then $P(y;z) = P(x;z)$

\[
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 4 \\
\end{bmatrix}, \quad z = (5,6,4), \quad \text{and} \quad g^0 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

*Example:* $x = \begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 4 \\
\end{bmatrix}$, $z = (5,6,4)$, and $g^0 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}$

Person 3 is not multidimensionally poor, does it matter if he/she experiences an increase in any of the dimensions?
Natural Extensions

**Deprivation Focus (Type II):** If $y$ is obtained from $x$ by a increment in achievements among the non-deprived, then $P(X;z) = P(Y;z)$. [Recall Deprived vs. Poor]

\[
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 4
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Example: $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 4 \end{bmatrix}$, $z = (5,6,4)$, and $g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Suppose person 2 is considered multidimensionally poor, does it matter if he/she experiences an increment in the third dimension in which he/she is not deprived?
Natural Extensions

Focus Axioms and Types of Identification

Each of the two focus axioms is attributed to each identification technique introduced earlier:

- Poverty focus is attributed to the Aggregated Poverty Line Approach
- Deprivation focus is attributed to the Counting Approach
Natural Extensions

**Continuity:** For any sequence $x$, if $x'$ converges to $x$, then $P(x';z)$ converges to $P(x;z)$

A technical assumption. It prevents poverty measures from changing abruptly for changes in distribution of achievements.

Similar intuitive interpretation as the assumption in single dimensional framework.
Natural Extensions

**Monotonicity:** There are, unlike in unidimensional framework, two types of monotonicity axiom

*(Type I)* Becoming less deprived in a specific dimension (within dimension):

*(Type II)* Becoming deprived in one less dimension (across dimensions): Dimensional Monotonicity
Natural Extensions

**Monotonicity:** If $y$ is obtained from $x$ by a *deprived increment* among the poor and the poverty line remains unchanged, then $P(y, z) < P(x, z)$

$y$ is obtained from $x$ by a *deprived increment* if there is an increment in a deprived achievement of a multidimensionally poor

$\begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$, $z = (5, 6, 4)$, $y = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$

**Example:** $x = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$, $z = (5, 6, 4)$, $y = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$

Person 1 is multidimensionally poor, and experiences an improvement in the **third dimension**.
Natural Extensions

**Dimensional Monotonicity:** If \( y \) is obtained from \( x \) by a *dimensional increment among the poor*, then \( P(y,z) < P(x,z) \)

\( y \) is obtained from \( x \) by a *dimensional increment among the poor* if due to an increment in a deprived achievement of a poor, he or she becomes non-deprived in that dimension.

\[
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3
\end{bmatrix}
\]

**Example:** \( x = \begin{bmatrix} 4 & 4 & 2 \end{bmatrix}, \ z = (5 \ 6 \ 4), \ y = \begin{bmatrix} 3 & 6 & 4 \end{bmatrix} \)

Suppose person 2 is considered multidimensionally poor, and experiences an increment in the second dimension and is no longer deprived in it.
Natural Extensions

Population Subgroups

Suppose the population size of x is denoted by n(x). Matrix x is divided into two population subgroups: x' with population size n(x') and x'' with population size n(x'') such that n(x) = n(x') + n(x'')

\[
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3
\end{bmatrix}
\]

Income  Education  Health

Person 1  Person 2  Person 3
Natural Extensions

**Population Subgroup Consistency:** If \( P(y';z) > P(x';z) \) and \( P(y'';z) = P(x'';z) \), and \( n(x') = n(y') \), \( n(y'') = n(x'') \), then \( P(y;z) > P(x;z) \)

**Population Subgroup Decomposability:** A poverty measure is additive decomposable if:

\[
P(x) = \frac{n(x')}{n} P(x') + \frac{n(x'')}{n} P(x'')
\]

Recall: *decomposability implies subgroup consistency, but the converse does not hold*
Analogous Concept: Dimensional Subgroups

\[ \mathbf{x} = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \]

**Person 1**

**Person 2**

**Person 3**

**Decomposability Across Dimensions**

It is a purely multidimensional concept, where the overall poverty can be expressed as an weighted average of dimensional deprivations (among poor only)
Natural Extensions

Transfer in unidimensional context: If $y$ is obtained from $x$ by a progressive transfer among the poor, then $P(y;z) < P(x;z)$

Recall if income is transferred from a person to another who is not richer than the former, keeping mean income same, the transfer is called a progressive transfer.

This is also known as Pigou-Dalton transfer principle.

Example: $z = 10$, $x = (9,4,15,8)$; $y = (9,5,15,7)$
Natural Extensions

**Transfer in multidimensional context:**

**Bistochastic matrix (B):** A matrix whose row elements and column element sum up to one

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

*Example:* A general bistochastic matrix

Multiply a vector by a bistochastic matrix

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 16 \end{bmatrix} = \begin{bmatrix} 7.6 \\ 8.8 \\ 11.6 \end{bmatrix}$$
Natural Extensions

Transfer in multidimensional context:

**Bistochastic matrix (B):** A matrix whose row elements and column element sum up to one

*Example:* What bistochastic matrix is used to obtain \( y = (9, 5, 15, 7) \) from \( x = (9, 4, 15, 8) \)?

It is \( B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0.25 \\ 0 & 0 & 1 & 0 \\ 0 & 0.25 & 0 & 0.75 \end{bmatrix} \)
Natural Extensions

**Uniform Majorization (UM):** Matrix $y$ is obtained from $x$ by a *Uniform Majorization among the poor* (an averaging of achievements among the poor) if $y = Bx$, where $B$ is an $n \times n$ bistochastic matrix but not a permutation matrix, and $b_{ii}=1$ for every non-poor person $i$ in $Y$.

$$
X = BY = \begin{bmatrix}
0.5 & 0.5 & 0 \\
0.5 & 0.5 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3
\end{bmatrix}
= \begin{bmatrix}
3.5 & 4.5 & 3 \\
3.4 & 4.5 & 3 \\
8 & 6 & 3
\end{bmatrix}, \text{ and } z = [5 \ 6 \ 5]
$$

Achievements of the first two persons (poor) were smoothed
Natural Extensions

*Transfer Under UM:* If \( y \) is obtained from \( x \) by a *uniform majorization among the poor* (an averaging of achievements among the poor), then \( P(y;z) \leq P(x;z) \).
Axiom Specific to the Multidimensional Case

Consider the following two matrices

\[
\begin{bmatrix}
7 & 7 & 2 \\
3 & 3 & 8 \\
1 & 0 & 1 & 0 & 1 & 2
\end{bmatrix}
\quad \text{Person 1}
\]

\[
\begin{bmatrix}
7 & 7 & 8 \\
3 & 3 & 2 \\
1 & 0 & 1 & 0 & 1 & 2
\end{bmatrix}
\quad \text{Person 1}
\]

\[
\begin{bmatrix}
7 & 7 & 8 \\
3 & 3 & 2 \\
1 & 0 & 1 & 0 & 1 & 2
\end{bmatrix}
\quad \text{Person 2}
\]

\[
\begin{bmatrix}
4 & 5 & 3 \\
\end{bmatrix}
\quad \text{Person 3}
\]

Is the pattern of poverty same in both societies?
If not, what is the difference?
Axiom Specific to the Multidimensional Case

Both matrices have the same distribution for each dimension (\textit{marginal distribution})

The correlation between dimensions are not same

Require an axiom based on correlation between dimension when marginals are same (Atkinson & Bourguignon, 1982; Boland & Proschan, 1988).

This axiom is intrinsic to the multivariate case
Axiom Specific to the Multidimensional Case

\[
\begin{bmatrix}
7 & 7 & 2 \\
3 & 3 & 8 \\
10 & 10 & 12
\end{bmatrix}
\]

\[
\begin{bmatrix}
7 & 7 & 8 \\
3 & 3 & 2 \\
10 & 10 & 12
\end{bmatrix}
\]

Ways to call the data transformation:

From x to y: association increasing rearrangement (Boland & Proschan, 1988); correlation-increasing transfer (Tsui, 1999), correlation increasing switch (Bourguignon & Chakravarty, 2003 and Chakravarty, 2010)

Axiom Specific to the Multidimensional Case

Matrix x is obtained from y by an association decreasing rearrangement among the poor if for two persons i and i',

i) Person i and person i' are poor in y

ii) In y, in no dimension, person i' has more achievement than person i

iii) In x, i and i' switch some of their achievements in such a way that i has more in some dimension and i' has more in some other dimensions (marginal distributions remain same)

iii) $y_{i''} = x_{i''}$ for all except i and i' or the amount of attributes of all other persons $i'' \neq i,i'$ remain unchanged

iv) Thus, $y_{i'}$ and $y_{i'}$ are comparable by vector dominance but not $x_{i'}$ and $x_{i'}$. 
Axiom Specific to the Multidimensional Case

Vector Dominance

Vector dominance between vector $b = (b_1, \ldots, b_d)$ and vector $c = (c_1, \ldots, c_d)$ occurs when $b_i \geq c_i$ for all $i$ and $b_i > c_i$ for some $i$

Vector dominance in $y$ for all three rows, but not in $x$

$$x = \begin{bmatrix} 7 & 7 & 2 \\ 3 & 3 & 8 \\ 10 & 10 & 12 \end{bmatrix} \quad y = \begin{bmatrix} 7 & 7 & 8 \\ 3 & 3 & 2 \\ 10 & 10 & 12 \end{bmatrix}$$
Question…

• How do you think poverty should change under an association decreasing rearrangement?
Association Decreasing Rearrangement

\[
\begin{bmatrix}
7 & 7 & 2 \\
3 & 3 & 8 \\
10 & 10 & 12 \\
\end{bmatrix}
\begin{bmatrix}
7 & 7 & 8 \\
3 & 3 & 2 \\
10 & 10 & 12 \\
\end{bmatrix}
\]

If you think that good health can substitute (compensate) for bad income or bad education, then poverty should decrease.

If you think that good health is necessary (complementary) to achieve good income and good education, then poverty should increase.

If you think that health is not necessary to achieve good income and good education, and can not either substitute for any of these, (i.e., you think they are independent), then poverty should not change.
Axiom Specific to the Multidimensional Case

**Weak Rearrangement:** If $x$ is obtained from $y$ by an association decreasing rearrangement among the poor, $P(y;z) \leq P(x;z)$

What is the assumption behind the axiom?

Assumption: Attributes are *independent* (if $=$) or *substitutes* (if $<$) (compensating achievements)

Could be *complements*, then axiom should go in the other way ($>$). (Bourguignon & Chakravarty, 2003).