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HDCP-IRC
The Human Development, Capability and International Research Centre
Instituto Universitario di Studi Superiori
www.iusspavia.it

OPHI
Oxford Poverty & Human Development Initiative
University of Oxford
www.ophi.org.uk
Robustness of Single-dimensional Inequality and Poverty Measures

Suman Seth
Vanderbilt University
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Introduction

• What is a Robust Comparison?
  – When a particular comparison is unambiguous

• Comparison of what?
  – Inequality
  – Poverty

• Focus of this lecture
  – To learn when any two distributions are comparable and to what extent
Robust Inequality Comparison
Two Classes of Measures

• Class of Lorenz Consistent Inequality Measures

• Class of Transfer Sensitive Inequality Measures
Lorenz Consistent Measures

• Any relative inequality measure satisfying four properties – *Symmetry, Replication Invariance, Scale Invariance* and *Pigou-Dalton Transfer Principle*, is Lorenz Consistent.
Comparing Two Distributions

• Example: Suppose there are two distributions

\[ X = \{10, 20, 30, 40\} \text{ and } Y = \{10, 25, 25, 40\} \]

• Both Distributions have the same mean.

• Which distribution is more equal?
Comparing Two Distributions

• How do the Lorenz curves look?

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<th>%Cum. Income $L(X, p)$</th>
<th>%Cum. Income $L(Y, p)$</th>
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<td>Population $(p)$</td>
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Comparing Two Distributions

• How the Lorenz curves would look like?
Comparing Two Distributions

• Any relative inequality measure will judge \( y \) as more equal compared to \( x \).

• Comparison of \( x \) and \( y \) is completely robust for all relative inequality measures.

• Consider the following pairs of distributions

\[
X = \{10, 20, 30, 40\} \text{ and } Y' = \{17, 17, 17, 49\} \\
X = \{10, 20, 30, 40\} \text{ and } Y'' = \{19, 19, 19, 43\}
\]
Comparing Two Distributions

• How do the Lorenz curves look?

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Comparing Two Distributions

- How the Lorenz curves would look like?

Which distribution is more equal: $X$ or $Y'$?

Gini($X$) = 0.25
Gini($Y'$) = 0.24
CV²($X$) = 0.27
CV²($Y'$) = 0.41
Comparison is not robust
Transfer Sensitive Measures

**Theorem.** (Shorrocks & Foster 1987) Suppose $X$ and $Y$ have the same mean and the Lorenz curve of $Y$ intersects that of $X$ once from above. Then

$$I(X) > I(Y)$$

for all transfer sensitive inequality measures $I(\cdot)$ if and only if

$$\text{sd}^2(X) \geq \text{sd}^2(Y)$$
Corollary. (Shorrocks & Foster 1987) Suppose $X$ and $Y$ have positive mean and the Lorenz curve of $Y$ intersects that of $X$ once from above. Then

$I(X) > I(Y)$ for all inequality measures $I(\cdot)$ satisfying transfer sensitivity, scale invariance and replication invariance, if and only if

$$CV^2(X) \geq CV^2(Y)$$
Transfer Sensitive Measures

• Which distribution is more equal according to all transfer sensitive inequality indices?

\[
CV(X) = 0.27 \\
CV(Y') = 0.41 \\
CV(Y'') = 0.23
\]

\(Y''\) is more equal than \(X\).
Transfer Sensitive Measures

• Comparison of $X$ and $Y''$ is robust for all transfer sensitive inequality indices

• Comparison of $X$ and $Y'$ is still not robust

• Note: with same mean, Lorenz consistency is equivalent to second order stochastic dominance and Transfer sensitivity is equivalent to third order stochastic dominance
Robust Poverty Comparison
Robust Comparison

• When a particular poverty comparison is robust or unambiguous?

• Robust with respect to what?

• Robust with respect to the poverty line chosen.
FGT class of indices

- For this part, we focus on FGT class of indices:

\[
\text{FGT}_\alpha(x; z) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{z - x_i^*}{z} \right)^\alpha = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{g_i^*}{z} \right)^\alpha
\]

- When \( \alpha = 0 \), \( \text{FGT}_0 = H \) (Headcount Ratio)
- When \( \alpha = 1 \), \( \text{FGT}_1 = HI = P_1 \) (Per Capita Income Gap)
- When \( \alpha = 2 \), \( \text{FGT}_2 = P_2 \) (Per Capita Squared Poverty Gap)
Head Count Ratio

- Head Count Ratio: \( H = \frac{q}{n}. \)

- Given income distributions to be compared

\[ X = \{10, 20, 30, 40\}, \quad Y = \{15, 25, 45, 55\} \]

- Suppose the poverty line is 22

\[ H(X; 22) = 2/4, \quad H(Y; 22) = 1/4 \]
Head Count Ratio

- Probability distribution of income $P(\cdot)$

$X = \{10, 20, 30, 40\}$
$Y = \{15, 25, 45, 55\}$

- What happens if poverty line is 35 or 50?
- $H(X; 35) = 3/4, \; H(Y; 35) = 2/4$;
- $H(X; 50) = 4/4, \; H(Y; 50) = 3/4$
Head Count Ratio

• In terms of head count ratio $Y$ has no greater poverty than $X$ has for all $z$

• In terms of probability distribution, $P(y) \overset{\text{FSD}}{\sim} P(x)$ since $P(y \leq z) \leq P(x \leq z) \forall z$ and $<$ for some $z$

• (Foster & Shorrocks 1988) The poverty ordering $H$ is thus identical to FSD.

• The poverty dominance of $Y$ over $X$ is robust for all poverty lines $z$ if and only if $P(y) \overset{\text{FSD}}{\sim} P(x)$
Head Count Ratio (not robust)

- Suppose distribution \( Y \) becomes \{15, 25, 30, 35\}

\[ H(X; 22) = \frac{2}{4}, \quad H(Y; 22) = \frac{1}{4}; \]
\[ H(X; 36) = \frac{3}{4}, \quad H(Y; 36) = \frac{4}{4} \]

\( X = \{10, 20, 30, 40\} \)

This comparison is not robust for \( H \)

- What happens if poverty line is 22 or 36?
Per Capita Income Gap

• Per Capita Income Gap: \( P_1(X) = \sum_{i \in q(X)} \frac{(z - x_i)}{nz} \)

• Given income distributions to be compared
  \( X = \{10, 20, 30, 40\} \), \( Y = \{15, 25, 30, 35\} \)

• Suppose the poverty line is 22

• \( P_1(X; 22) = \frac{[(22-10) + (22-20)]}{(4 \times 22)} = 0.16 \)
• \( P_1(Y; 22) = \frac{(22-15)}{(4 \times 22)} = 0.08 \)
Per Capita Income Gap

- \( P_1(X; 22) = (A + B + C)/22 \), \( I(Y; 22) = B/22 \)

- If the poverty line is \( z = 45 \)
  - \( P_1(X; 22) = (A + B + C + D + E + F)/22 = 0.44 \)
  - \( P_1(Y; 22) = (B + E + F)/22 = 0.42 \)
Per Capita Income Gap

• In terms of per capita income gap, $Y$ has no greater poverty than $X$ for all $z$

• In terms of probability distribution, $P(y)$ SSD $P(x)$ since $\int z P(y \leq s) ds \geq \int z P(x \leq s) ds \forall z$ and $<$ for some $z$.

• (Foster & Shorrocks 1988) The poverty ordering $P_1$ is thus identical to SSD.

• Dominance of $Y$ over $X$ is robust in terms of per capita income gap if and only if $P(y)$ SSD $P(x)$
Per Capita Squared Income Gap

• Per Capita Income Gap is not sensitive to transfer

• Per Capita Squared Income Gap is defined by \( P_2(X) = \sum_{i \in q(X)} (z - x_i)^2/nz^2 \)

• In terms of probability distribution, \( P(y) \) TSD \( P(x) \) if and only if \( \exists z \exists t P(y \leq s) ds dt \leq \exists z \exists t P(x \leq s) ds dt \) \( \forall z \) and \( < \) for some \( z \).
Per Capita Squared Income Gap

• (Foster & Shorrocks 1988) The poverty ordering $P_2$ is identical to TSD.

• Dominance of $Y$ over $X$ is robust in terms of per capita squared income gap if and only if $P(y) \text{TSD} P(x)$
Stochastic Dominance Rules

• An easy way to check different degrees of stochastic dominance

• Vectors: \( X = \{10, 20, 30, 40\} \), \( Y = \{15, 25, 45, 55\} \)

• First, arrange them in ascending order (if they are not already)
Stochastic Dominance Rules

• First Degree
  – Check each element of the vectors
  – If $y_i \geq x_i$ for all $i$, then $Y$ FSD $X$

• Second Degree
  – Construct $X'$ and $Y'$ such that $x_1' = x_1$, $y_1' = y_1$, and
    $x_i' = x_i + x_{i-1}$, $y_i' = y_i + y_{i-1}$ for $i = 2, 3, \ldots$
Stochastic Dominance Rules

- **Second Degree**
  - \( X = \{10, 20, 30, 40\} \), \( Y = \{15, 25, 45, 55\} \)
  - \( X' = \{10, 30, 60, 100\} \) and \( Y' = \{15, 40, 85, 140\} \)
  - If \( y'_i \geq x'_i \) for all \( i \), then \( Y \) SSD \( X \)

- **Third Degree**
  - Construct \( X'' \) and \( Y'' \) such that \( x''_1 = x'_1 \), \( y''_1 = y'_1 \), and \( x''_i = x'_i + x'_{i-1} \), \( y''_i = y'_i + y'_{i-1} \) for \( i = 2,3,... \)
Stochastic Dominance Rules

• Third Degree

- $X' = \{10, 30, 60, 100\}$ and $Y' = \{15, 40, 85, 140\}$
- $X'' = \{10, 40, 100, 200\}$ and $Y'' = \{15, 55, 140, 280\}$
- If $y''_i \geq x''_i$ for all $i$, then $Y$ TSD $X$