Inequality-adjusted HDI: the basics

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Introduction: What is the inequality-adjusted HDI

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- The IHDI is a generalization of the HDI that takes into account, for every dimension, both the average achievement and the way it is distributed across the population.
- The IHDI is based on the measures of Foster, Lopez-Calva and Szekely (2005), which in turn are based on the inequality indices of Atkinson (1970).
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- The IHDI is a generalization of the HDI that takes into account, for every dimension, both the average achievement and the way it is distributed across the population.
- The IHDI is based on the measures of Foster, Lopez-Calva and Szekely (2005), which in turn are based on the inequality indices of Atkinson (1970).
- Since the IHDI is never higher than the HDI, the former is interpreted as "actual human development", penalized by inequality, while the latter means "potential human development" (should inequality be completely suppressed).
The basic formulas of the IHDI: The Atkinson statistic for the geometric mean

When the inequality aversion parameter, $\epsilon$, is equal to 1, the statistic $A_x$ is:

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A_x = 1 - \frac{\prod_{n=1}^{N} x_n}{\sqrt[N]{\sum_{n=1}^{N} x_n}}
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When the inequality aversion parameter, $\epsilon$, is equal to 1, the statistic $A_x$ is:

$$A_x = 1 - \frac{\prod_{n=1}^{N} x_n^{\frac{1}{N}}}{\bar{x}} = 1 - \frac{\sqrt[N]{x_1 \cdots x_N}}{\bar{x}}$$

Where: $\bar{x} = \frac{1}{N} \sum_{n=1}^{N} x_n$
The basic formulas of the IHDI: Relationship between mean attainment and inequality

The geometric mean penalizes the arithmetic mean by the degree of inequality:

$$\sqrt[N]{x_1 \cdots x_N} = \bar{x}(1 - A_x)$$
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The geometric mean penalizes the arithmetic mean by the degree of inequality:

\[ \frac{1}{N} \prod_{i=1}^{N} x_i = \bar{x}(1 - A_x) = \sqrt[\sqrt{\cdots\sqrt{x_1 \cdots x_N}}] \]

Similarly, the inequality-adjusted dimension index is obtained from the HDI dimension index, by penalizing it with the inequality loss:

\[ I_{I_x} = I_x(1 - A_x) \]
The basic formulas of the IHDI: Computing the IHDI

Then the dimension-specific, inequality-adjusted measures are aggregated using, again, the geometric mean.

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In the specific case of the HDR:

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In both cases, the IHDI and the HDI, are sensitive to inequality across the normalized dimensions. They penalize ”unbalanced relative development”.
Notice the following interesting relationship between the IHDI and the HDI:

\[ IHDI = \sqrt[3]{(1 - A_{Life})(1 - A_{Education})(1 - A_{Income})} \times HDI \]
The basic formulas of the IHDI: the loss

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The HDI represents the highest possible level, the "potential", that the IHDI could get if one could freely transfer achievements across people, in order to eliminate within-dimension inequality.
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$$IHDI = \sqrt[3]{(1 - A_{\text{Life}})(1 - A_{\text{Education}})(1 - A_{\text{Income}})} \times HDI$$

The HDI represents the highest possible level, the ”potential”, that the IHDI could get if one could freely transfer achievements across people, in order to eliminate within-dimension inequality. From this relationship we can compute the percentage loss in ”potential HDI” due to inequality:

$$Loss = 1 - \frac{IHDI}{HDI} = 1 - \sqrt[3]{(1 - A_{\text{Life}})(1 - A_{\text{Education}})(1 - A_{\text{Income}})}$$
The basic formulas of the IHDI: Example

**Example: Slovenia**

<table>
<thead>
<tr>
<th>Indicator</th>
<th>Dimension index</th>
<th>Inequality measure (A1)</th>
<th>Inequality-adjusted index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Life expectancy</td>
<td>78.8</td>
<td>0.930</td>
<td>0.043</td>
</tr>
<tr>
<td>Mean years of schooling</td>
<td>9</td>
<td>0.682</td>
<td></td>
</tr>
<tr>
<td>Expected years of schooling</td>
<td>16.7</td>
<td>0.811</td>
<td></td>
</tr>
<tr>
<td>Education index</td>
<td></td>
<td>0.782</td>
<td>0.040</td>
</tr>
<tr>
<td>Logarithm of GNI</td>
<td>10.16</td>
<td>0.780</td>
<td></td>
</tr>
<tr>
<td>GNI</td>
<td>25,857</td>
<td>0.238</td>
<td>0.122</td>
</tr>
</tbody>
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**Human Development Index**

| HDI with unlogged income   | \( \sqrt[3]{0.930 \cdot 0.782 \cdot 0.238} = 0.557 \) | \( \sqrt[3]{0.890 \cdot 0.751 \cdot 0.209} = 0.519 \) | 1−0.519/0.557 = 0.068 |
| HDI                        | \( \sqrt[3]{0.930 \cdot 0.782 \cdot 0.780} = 0.828 \) | \( (0.519 / 0.557) \cdot 0.828 = 0.772 \) |

*Note: Values are rounded.*
The IHDI versus the HDI: an Example

**Figure 5.1** Inequality has large impacts on human development

Loss in HDI due to multidimensional inequality

Note: Numbers beside bars are percentage loss due to multidimensional inequality (see statistical table 5).
Source: HDR0 calculations using data from the HDR0 database.
The basic formulas of the IHDI: Decomposition of the loss

The Loss function can be decomposed into an approximation of the sum of the dimension’s inequality contributions by noticing that:

\[ \text{Loss} \approx \frac{1}{D} \sum_{d=1}^{D} A_d \]

Then the contribution of inequality in, say, education, can be computed as:

\[ C_{\text{Education}} = \frac{A_{\text{Education}}}{D} \sum_{d=1}^{D} A_d \]
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Decomposition of the loss: Example
The basic formulas of the IHDI: Data aspects and requirements

Q: What data do we need?

- Ideally, disaggregated at the individual level.
- Not necessary for the IHDI. Because it is not sensitive to the association of dimensions, there is no need to rely on the same dataset for all the variables.
- Inequality will be underestimated for that variable and then for the whole IHDI.
- Taking the log of income compresses inequality. It's better to work with the levels. However, for flexible options see the appendix of the HDR 2010.
- The IHDI cannot be computed sensibly with negative values and one zero suffices for $A \times 1 = A$
  Disposing of those observations or imputing values may be necessary. The HDR 2010 replaces them with the minimum value of the bottom 0.5 percentile distribution of positive incomes. It may be necessary to assess the impact of imputations on inequality.
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Q: What if I have, say, municipal data for some variable? A: Inequality will be underestimated for that variable and then for the whole IHDI.

Q: I want to use an income measure, shall I take its log? A: Taking the log of income compresses inequality. It’s better to work with the levels. However, for flexible options see the appendix of the HDR 2010.

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▶ Path-independence.
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- Sensitivity to progressive rank-preserving transfers within dimensions. The IHDI increases when a richer person transfers part of its achievement to a poorer person, and their pairwise ranks do not change.

- Path-independence. If $\overline{x} = l_x$ then the IHDI would also be path independent, i.e. $IHDI = \prod_{d=1}^{D} \prod_{n=1}^{N} x_{id}^{\frac{1}{ND}}$
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- Path-independence. If $\bar{x} = I_x$ then the IHDI would also be path independent, i.e. $IHDI = \prod_{d=1}^{D} \prod_{n=1}^{N_d} x_{id}^1$.

- Sub-group consistency. Notice that: $\bar{x}(1 - A_x) = \prod_{j=1}^{G} [\prod_{i=1}^{N_j} x_i^{1/N_j}]^{N_j/N}$.
Some properties of the IHDI:

- Individual scale invariance.
- Independence of standardized values.
- Consistency over time.

Multiplying one variable by a constant does not change $1 - A_x$, although the IHDI gets re-scaled, yet rankings are preserved, percentage differences are preserved, and even the Loss function remains unchanged.

Standarization (i.e. division of the dimensional achievement by a common value across countries, e.g. that of one specific country) does not affect country rankings.

Since scale revisions do not affect relative and percentage rankings, then these remain consistent over time (i.e. they do not themselves constitute a source of change).
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Some interpretation and policy aspects of IHDI

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- The IHDI should not be compared to a Gini coefficient. Discounting the mean attainment with a Gini coefficient, $\mu(x)[1 - G(x)]$, has been done (Anand and Sen, 1993; Hicks, 1997), but, unlike the FLS measures, the Gini is generally not sub-group consistent and it is not path-independent.
Limitations of the IHDI

- Lack of association sensitivity.
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- Zero and negative values.