THE MEASUREMENT OF THE INEQUALITY OF INCOMES

1. It is generally agreed that, other things being equal, a considerable reduction in the inequality of incomes found in most modern communities would be desirable. But it is not generally agreed how this inequality should be measured. The problem of the measurement of the inequality of incomes has not been much considered by English economists. It has attracted rather more attention in America, but it is in Italy that it has hitherto been most fully discussed. The importance of the problem has been obscured by the inadequacy of the available statistics of the distribution of income in all modern communities. To such statistics as we have, no very fine measures can be applied. The improvement of these statistics is the business of statisticians, but the problem of measuring and comparing the inequalities, which improved statistics would more precisely reveal, should be capable of theoretical solution now. No complete solution is presented in this paper, but only a discussion of certain points of principle and method.

2. First, as to the nature of the problem. An American writer has expressed the view that "the statistical problem before the economist in determining upon a measure of the inequality in the distribution of wealth is identical with that of the biologist in determining upon a measure of the inequality in the distribution of any physical characteristic."[1] But this is clearly wrong. For the economist is primarily interested, not in the distribution of income as such, but in the effects of the distribution of income upon the distribution and total amount of economic welfare, which may be derived from income. We have to deal, therefore, not merely with one variable, but with two, or possibly more, between which certain functional relations may be presumed to exist.

A partial analogy would be found in the problem of measuring the inequality of rainfall in the various districts of a large agricultural area. From the point of view of the cultivator, what is

[1] Persons, *Quarterly Journal of Economics*, 1908–9, p. 431.

important is not rainfall as such, but the effects of rainfall upon the crop which may be raised from the land. Between rainfall and crop there will be a certain relation, the discovery of which will be a matter of practical importance. The objection to great inequality of rainfall is the resulting loss of potential crop. The objection to great inequality of incomes is the resulting loss of potential economic welfare.

Let us assume, as is reasonable in a preliminary discussion, that the economic welfare of different persons is additive, that the relation of income to economic welfare is the same for all members of the community, and that, for each individual, marginal economic welfare diminishes as income increases. Then, if a given income is to be distributed among a given number of persons, it is evident that economic welfare will be a maximum, when all incomes are equal. It follows that the inequality of any given distribution may conveniently be defined as the ratio of the total economic welfare attainable under an equal distribution to the total economic welfare attained under the given distribution. This ratio is equal to unity for an equal distribution, and is greater than unity for all unequal distributions. It may, therefore, be preferred to define inequality as this ratio minus unity, but for comparative purposes this modification of the definition is unnecessary. Inequality, however, though it may be *defined* in terms of economic welfare, must be *measured* in terms of income.

3. Starting from the above definition, it is clear that, if we assume any precise functional relation between income and economic welfare, we can deduce a corresponding measure of inequality. It is also clear that, under this procedure, no *one* measure of inequality will emerge, whose appropriateness will be independent of the particular functional relation assumed.

The procedure suggested may be illustrated by two examples. Take, first, the hypothesis that proportionate additions to income, in excess of that required for "bare subsistence," make equal additions to economic welfare. This is Bernoulli's hypothesis, except that economic welfare is substituted for satisfaction.[1]

Then, if w = economic welfare and x = income, we have—

$$dw = \frac{dx}{x},$$

or

$$w = \log x + c.$$

If $x_1, x_2, \ldots x_n$ are individual incomes, whose arithmetic mean

[1] A discussion of the distinction, if any, between economic welfare and satisfaction lies outside the scope of this paper.

is x_a and geometric mean x_g, the corresponding measure of inequality is, by our definition—

$$\frac{n \log x_a + nc}{n \log x_g + nc} = \frac{\log x_a + c}{\log x_g + c}.$$

If we assume that, when $x=1$, $w=0$, then $c=0$, and our measure of inequality becomes $\frac{\log x_a}{\log x_g}$. It may, at first sight, be thought that a still simpler, and practically equivalent, measure will be $\frac{x_a}{x_g}$, but this simplification raises a question to which further reference will be made below.

The above hypothesis, however, is not satisfactory. Apart from the difficulty that only income in excess of that required for "bare subsistence" is taken into account, it is clear that too rapid a rate of increase of economic welfare is assumed, when income becomes large. After a certain point it is pretty obvious that more than proportionate additions to income will generally be required, in order to make equal additions to economic welfare. To be even tolerably realistic, a formula connecting income with economic welfare should satisfy the following conditions.

(1) Equal increases in economic welfare, at any rate after income is greater than a certain amount, should correspond to more than proportionate increases in income; (2) economic welfare should tend to a finite limit, as income increases indefinitely; (3) economic welfare should be zero for a certain amount of income, and negative for smaller amounts. These conditions are satisfied, if we assume that the relation of economic welfare to income is of the form $dw = \frac{dx}{x^2}$, so that $w = c - \frac{1}{x}$, where c is a constant. For then, however large x becomes, w can never become larger than c, and, when x is less than $\frac{1}{c}$, w is negative.[1] If we adopt this formula, which appears to be a good compromise of its kind between plausibility and simplicity, the corresponding measure of inequality is—

$$\frac{nc - \frac{n}{x_a}}{nc - \frac{n}{x_h}} = \frac{c - \frac{1}{x_a}}{c - \frac{1}{x_h}}$$

where x_h is the harmonic mean of the individual incomes, and

[1] If it were practicable to fix a unit of economic welfare, it would have to be fixed, in relation to the unit of income, so that both these attributes of c would hold good. There is no theoretical objection to this.

c, as already stated, the reciprocal of the minimum income, which yields positive economic welfare.

Both the measures of inequality obtained above are simple in form and have a certain theoretical elegance. But neither is readily applicable to statistics. The arithmetic mean is, indeed, easily calculated from perfect statistics, and fairly easily approximated to from imperfect statistics, but the corresponding calculations for the geometric and harmonic means are very laborious, when the number of individual incomes is large, and the corresponding approximations, especially for the harmonic mean, are practically impossible, where the statistics show more than a small degree of imperfection. The first of the two measures, moreover, involves an estimate of the income necessary for "bare subsistence," and the second an estimate of the minimum income which yields positive economic welfare. And neither of these estimates are easily made. Nor, of course, have we really any precise knowledge of the functional relation between income and economic welfare.

4. Failing such precise knowledge, we may still lay down certain general principles, which shall serve as tests, to which various plausible measures of inequality may be submitted. We have, first, what may be called the principle of transfers. Maintaining the assumptions laid down in Section 2 above, we may safely say that, if there are only two income-receivers, and a transfer of income takes place from the richer to the poorer, inequality is diminished.[1] There is, indeed, an obvious limiting condition. For the transfer must not be so large, as more than to reverse the relative positions of the two income-receivers, and it will produce its maximum result, that is to say, create equality, when it is equal to half the difference between the two incomes. And we may safely go further and say that, however great the number of income-receivers and whatever the amount of their incomes, any transfer between any two of them, or, in general, any series of such transfers, subject to the above condition, will diminish inequality.[2] It is possible that, in comparing two distributions, in which both the total income and the number of income-receivers are the same, we may see that one might be able to be evolved from the other by means of a series of transfers

[1] Compare Pigou, *Wealth and Welfare*, p. 24.

[2] Inequality is certain to be diminished by a series of transfers such that all transfers from A, the richer, to B, the poorer, still leave A richer than, or just as rich as, B. But if some of the transfers make B richer than A, t is possible that the effects of the series of transfers might cancel out and eave the inequality the same as before.

of this kind. In such a case we could say with certainty that the inequality of the one was less than that of the other.

5. Let us now apply the principle of transfers to various measures of dispersion used by statisticians for measuring inequality in general. A distinction may be drawn between measures of relative dispersion and measures of absolute dispersion. Measures of relative dispersion will be simply numbers, while measures of absolute dispersion will be, in the present case, numbers of units of income. Most of the general measures of dispersion proposed by statisticians are measures of absolute dispersion, but are easily transformed into measures of relative dispersion, when divided by an appropriate divisor.

Consider first the mean deviation from the arithmetic mean. This measure is the sum of two parts, one of which comprises the deviations above, the other the deviations below, the mean.[1] It is a bad measure, judged by the principle of transfers, for it is unaffected by transfers within either part, provided that no income previously above the mean is reduced below it, and conversely. The transfer of a given sum from incomes above the mean to incomes below it, as, for example, by the provision of old age pensions for persons of small incomes from the proceeds of a tax on large incomes, would obviously reduce the mean deviation. But it would be unaffected, if such pensions were provided by a tax levied on those whose incomes were just below the mean, or if additional comforts for millionaires were provided from a tax on those whose incomes were just above the mean, provided that none of the latter were reduced below the mean by the tax.

The mean deviation is a measure of absolute dispersion. If we divide it by the arithmetic mean, we obtain what we may call the relative mean deviation, which is equally insensitive to transfers wholly above or wholly below the mean.

Consider next the standard, or mean square, deviation from the arithmetic mean, *i.e.*, the square root of the arithmetic average of the squares of deviations from the arithmetic mean. The standard deviation is perfectly sensitive to transfers,[2] and thus passes our first test with distinction. Dividing the standard

[1] Thus if S_1 is the sum of the deviations of incomes greater than the mean and S_2 the sum of the deviations of incomes less than the mean, the mean deviation $= \frac{1}{n}(S_1+S_2)$, where n is the total number of incomes.

[2] For, if δ be the initial standard deviation of any distribution of n incomes, and δ' the standard deviation after an amount h has been transferred from an income x_1 to an income x_2, all other incomes remaining the same, we have $n^2(\delta^2-\delta'^2)=2h(x_1-x_2)-2h^2$. Therefore $\delta=\delta'$, only if $h=o$ or if $h=x_1-x_2$.

deviation by the arithmetic mean, we obtain what may be called the relative standard deviation. This, too, is perfectly sensitive to transfers.

Consider next Professor Bowley's quartile measure of dispersion, $\frac{Q_3 - Q_1}{Q_3 + Q_1}$, where Q_1 and Q_3 are quartiles.[1] This is a measure of relative dispersion. It is sensitive to transfers, in so far as these involve movements of the quartiles, but not otherwise. In this respect it is somewhat more sensitive than the mean deviation, but much less sensitive than the standard deviation.

An interesting measure of dispersion, which has not, I think, hitherto attracted the attention of English writers, is Professor Gini's mean difference, which, as applied to incomes, is the arithmetic average of the differences, taken positively, between all possible pairs of incomes.[2] It may be shown that this mean difference is equal to the weighted arithmetic mean of deviations from the median, the weights being proportionate to the number of incomes, increased by one, which are intermediate in size between the median and the income whose deviation is being considered.[3] The mean difference, thus defined, is a measure of absolute dispersion. Dividing it by the arithmetic mean, we obtain a measure of relative dispersion, which may be called the relative mean difference. The mean difference, whether absolute or relative, is perfectly sensitive to transfers.

Another interesting measure of inequality is based upon what some writers have called a Lorenz curve.[4] (See next page.) This is a simple and convenient graphical method of exhibiting any distribution of income, provided that our interest is confined to proportions, rather than absolute amounts, both of total income and of the number of income-receivers.

Along the axis Ox are measured percentages of the total income, and along the axis Oy the minimum percentages of the total number of income-receivers, who receive various percentages of the total income. For example, if the richest 20 per cent. of the population received 75 per cent. of the total income, this fact would determine one point ($x=75$, $y=20$), upon the Lorenz curve. A perfectly equal distribution

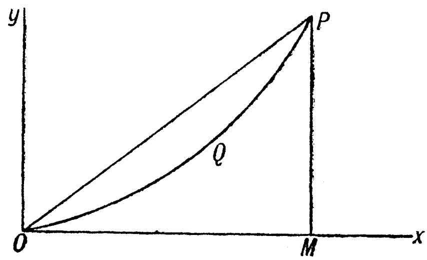
[1] Compare Bowley, *Elements of Statistics*, p. 136.

[2] See, for a discussion of this measure, Gini, *Variabilità e Mutabilità*.

[3] Gini, *ibid.*, pp. 32–33.

[4] Originally proposed by Mr. M. O. Lorenz, *Publications of the American Statistical Association* (1907), Vol. ix, pp. 209 ff. M. Séailles also recommended it, apparently independently, in 1910 in his *Répartition des Fortunes en France*, pp. 56–7. Sir Leo Chiozza Money had already given hints of this measure in his *Riches and Poverty*, first published in 1905.

would be represented by the straight line OP inclined at an angle of 45° to either axis. An unequal distribution would be represented by a curve, such as OQP, lying below the line OP. If MP is perpendicular to OM, $OM = MP = 100$, and an obvious measure of inequality is the area enclosed between the Lorenz curve and the line of equal distribution OP. The larger this area, the larger the inequality.



A remarkable relation has been established between this measure of inequality and the relative mean difference, the former measure being always equal to half the latter.[1]

Something will be said below concerning Professor Pareto's well-known measure of the inequality of incomes. But this measure cannot be tested, with reference to the principle of transfers, since it is based upon a supposed law, according to which, if the total income and the number of income-receivers are known, the distribution is uniquely determined.

6. So far, then, as tested by the principle of transfers, the standard deviation, whether absolute or relative, and the mean difference, whether absolute or relative, are good measures; Professor Bowley's quartile measure is a very indifferent measure; the mean deviation, whether absolute or relative, is a bad measure; and Professor Pareto's measure evades judgment. But the scope of the principle of transfers, as a test of measures of inequality, is narrowly limited. It can only be applied to some cases—and by no means to all—in which both the total income and the number of income-receivers are constant, and distribution varies.[2]

[1] For a mathematical proof of this see Ricci, *L'Indice di Variabilità*, pp. 22–24. The proof was first given, apparently, by Professor Gini. Another most elegant proposition, due to Professor Ricci (*ibid.*, pp. 32–33), is that, if any straight line be drawn parallel to the line of equal distribution, then all the Lorenz curves, to which this straight line is a tangent, represent distributions having the same relative mean deviation.

[2] Professor Pigou (*Wealth and Welfare*, p. 25 n.) uses the following argument to prove that, in these circumstances, a reduction in the standard deviation will probably increase aggregate satisfaction. "If A be the mean income and

It cannot be applied when either the total income or the number of income-receivers varies, or when both vary simultaneously. For these more general cases further tests are required, and three general principles suggest themselves as serviceable for this purpose.

7. We have, first, what may be called the principle of proportionate additions to incomes. It is sometimes suggested that proportionate additions to, or subtractions from, all incomes will leave inequality unaffected.[1] But, if the definition of inequality given above be accepted, this is not so. It appears, rather, that proportionate additions to all incomes diminish inequality, and that proportionate subtractions increase it. This is the principle of proportionate additions to incomes just referred to. A general proof of this principle presents difficulties, and is not attempted here, but the proof in two important special cases is easy. For, first, assume, using the same notation as in Section 3 above, that the relation of income to economic welfare is $w=\log x$. Then, if δ be the inequality of any given distribution, we have

$$\delta=\frac{\log x_a}{\log x_g}$$

Let all incomes be multiplied by θ and let δ' be the inequality of the new distribution.

a_1, a ,.... deviations from the mean, aggregate satisfaction, on our assumption

$$=nf(A)+(a_1+a_2+\ldots)f'+\frac{1}{2\,!}(a_1^2+a_2^2+\ldots.)f''+\frac{1}{3\,!}(a_1^3+a_2^3+\ldots.)f'''+\ldots.$$

But we know that $a_1+a_2+\ldots.=0$. We know nothing to suggest whether the sum of the terms beyond the third is positive or negative. If, therefore, the third and following terms are small relatively to the second term, it is certain, and, in general, it is probable that aggregate satisfaction is larger, the smaller is $(a_1^2+a_2^2+\ldots.)$. This latter sum, of course, varies in the same sense as the .... standard deviation." This argument would be strong, if all deviations were small, *i.e.* if inequality were already very small. But when, as is the case in all important modern communities, a number of the deviations are very large, it is quite likely that successive terms in the expansion will go on increasing (numerically) for some time, and this is specially likely as regards the series of alternate terms, which involve deviations raised to *even* powers. This likelihood will vary according to the form of the function f, but it seems clear that the third and following terms cannot, in general, be neglected. It follows that, in general, there is no certainty and only a somewhat low and problematical degree of probability, that a reduction in the standard deviation will increase satisfaction. There is no reason to suppose that it is not at least equally probable that a reduction in certain other measures of dispersion will have the same effect. One good test of the relative appropriateness of various measures of the inequality of incomes would be the relative probability that a reduction in such measures would increase economic welfare (or satisfaction), on the assumption that both the total income and the number of income receivers were constants. But the evaluation of such relative probabilities presents difficulties.

[1] See, *e.g.*, Taussig, *Principles o' Economics*, II, p. 485.

Then $\delta'=\frac{\log\theta+\log x_a}{\log\theta+\log x_g}$, and, since $x_a>x_g$, we have $\delta>\delta'$, if $\log\theta>0$, that is to say, if $\theta>1$.

Similarly, $\delta<\delta'$, if $\theta<1$.

That is to say, proportionate additions to all incomes diminish inequality and proportionate subtractions increase it.[1] This is true, if x is the total income of any individual. *A fortiori*, it is true, if x is surplus income in excess of "bare subsistence." For equal proportionate additions to surplus income involve larger proportionate additions to total income, when the latter is large, than when it is small. A series of transfers from richer to poorer will, therefore, transform proportionate additions to surplus incomes into proportionate additions to total incomes.

Next assume that the relation of income to economic welfare is $w=c-\frac{1}{x}$. Then, if δ be the inequality of any given distribution, we have $\delta=\dfrac{c-\frac{1}{x_a}}{c-\frac{1}{x_h}}$.

Let all incomes be multiplied by θ and let δ' be the inequality of the new distribution.

Then $\delta'=\dfrac{c-\frac{1}{\theta x_a}}{c-\frac{1}{\theta x_h}}$, and we have $\delta>\delta'$, if $(x_a-x_h)\ (\theta-1)>0$.

But $x_a>x_h$. $\therefore\ \delta>\delta'$, if $\theta>1$.

Similarly, $\delta<\delta'$, if $\theta<1$.

That is to say, proportionate additions to all incomes diminish inequality, and proportionate subtractions increase it.

8. If the principle of proportionate additions to incomes thus enunciated be provisionally accepted as true generally,[2] and not merely for the particular hypotheses just examined, a second principle follows as a corollary. This may be called the principle of equal additions to incomes, and is to the effect that equal additions to all incomes diminish inequality and equal subtractions increase it. Here, again, a direct general proof presents

[1] If we write $\delta=x_a/x_g$, instead of $\delta=\log x_a/\log x_g$, proportionate additions or subtractions will leave inequality unaffected. It follows that x_a/x_g is not a mere simplification of the measure $\log x_a/\log x_g$ arrived at in section 3 above, but is a distinct, and inferior, measure.

[2] The additions must, of course, be genuine. Inequality in this country would not be diminished by reckoning everyone's income in shillings, instead of in pounds. Units of money income in any two cases to be compared must have approximately equal purchasing power.

difficulties, though several writers have regarded the principle as so obvious that no proof is required.[1] But as a corollary of the preceding principle the proof is easy. For, let the total additional income involved in proportionate additions to all incomes be redistributed among income-receivers in such a way as to make equal, instead of proportionate, additions to all incomes. Then the addition to maximum economic welfare attainable is the same in both cases. But the addition to economic welfare actually attained is obviously greater, when additions to incomes are equal, than when they are proportionate. Therefore, inequality is smaller after equal additions have been made than after proportionate additions have been made, the total additional income being the same in both cases. But proportionate additions reduce inequality. Therefore, *a fortiori*, equal additions reduce inequality.[2]

9. The third principle may be called the principle of proportionate additions to persons, and is to the effect that inequality is unaffected if proportionate additions are made to the number of persons receiving incomes of any given amount. This, again, is easily proved. For the maximum economic welfare attainable and the economic welfare actually attained will both have been increased in the same proportion, and hence their ratio will be unaltered.

10. We may now test, by means of these three principles, the measures of inequality which have already been tested by means of the principle of transfers. Simple mathematical operations yield the following results :—

		Effect of	
Upon:	Proportionate Additions to Incomes.	Equal Additions to Incomes.	Proportionate Additions to Persons.
Absolute Mean Deviation.	Increased ...	Unchanged ...	Unchanged
Relative Mean Deviation.	Unchanged ...	Diminished ...	Unchanged
Absolute Standard Deviation. ...	Increased ...	Unchanged ...	Unchanged
Relative Standard Deviation. ...	Unchanged ...	Diminished ...	Unchanged
Bowley's Quartile Measure.	Unchanged ...	Diminished ...	Unchanged
Absolute Mean Difference.	Increased ...	Unchanged ...	Unchanged
Relative Mean Difference.	Unchanged ...	Diminished ...	Unchanged

Here the three absolute measures of dispersion give one set of identical results, and the four relative measures another. None

[1] "An equal addition to everyone's income . . . . . obviously makes incomes more equal than they would otherwise be." Cannan, *Elementary Political Economy*, p. 137. See also Loria, *La Sintesi Economica*, p. 369.

[2] Or alternatively, the total additional income being given, a distribution involving equal additions to all incomes may be evolved from a distribution involving proportionate additions to all incomes by means of a series of transfers from richer to poorer.

of the seven measures pass the test of proportionate additions to incomes, but the relative measures come nearer to doing so than the absolute measures.[1] The relative measures pass the test of equal additions to incomes, but the absolute measures do not. All seven measures pass the test of proportionate additions to persons. We may therefore eliminate the three absolute measures from further consideration. As between the four relative measures, the order of merit established by reference to the principle of transfers may stand, so far, unchanged, viz. :—

1 and 2. Relative standard deviation and relative mean difference (bracketed equal).
3. Bowley's quartile measure.
4. Relative mean deviation.

11. Can Professor Pareto's measure be brought into this order of merit? This is a relative measure, which is only applicable when distribution is approximately of the form $y=\frac{A}{x^a}$, where x is any income, y the number of incomes greater than x, and A and a constants for any given distribution, but variables for different distributions.[2] Assuming this formula for distribution, which, as Professor Bowley has shown,[3] is the same thing as assuming that the average of all incomes greater than x is proportional to x, Professor Pareto treats a as the measure of inequality, in the sense that, the greater a, the greater inequality. It follows mathematically that "neither an increase in the minimum income nor a diminution in the inequality of incomes can come about, except when the total income increases more rapidly than the population."[4] In other words, increased production per head is both a necessary condition and a sufficient guarantee of a diminution of inequality.

Professor Pareto's law, about which much has been written both by way of criticism and of qualified appreciation, implies a uniformity in distribution, which makes it impossible to apply either the principle of transfers or the principle of equal additions to incomes. Like the four other measures just considered, it is

[1] It should be noticed that, if we are comparing the inequality of two distributions by means of a measure which is unchanged by proportionate additions to incomes, it is not necessary that the unit of money income in the two distributions should have approximately the same purchasing power.

[2] Compare Pareto, *Cours d'Economie Politique*, II, pp. 305 ff, and *Manuel d'Economie Politique*, pp. 391 ff.

[3] *Measurement of Social Phenomena*, p. 106.

[4] *Cours*, II, pp. 320–1.

unchanged both by proportionate additions to incomes and by proportionate additions to persons. It has been suggested[1] that this measure, where it is applicable, will be in general accord with other plausible measures of dispersion. But, in view of the investigations of Italian economists, this is very doubtful.[2] It seems on the whole more likely, though the question requires further study, that, in order to bring it into general accord with other measures, the Pareto measure should be inverted, so that, the greater a, the smaller inequality. But such an inversion will explode Professor Pareto's alleged economic harmonies, and it will follow, according to his law, that increased production per head will always mean increased inequality!

According to Professor Gini,[3] many actual distributions of income approximate to the formula

$$n = \frac{1}{c}\, s^{\delta}, \text{ or } \log n = \delta \log s - \log c,$$

where s is the total income of the n richest income-receivers and δ and c are constants for any given distribution. He proposes δ as a measure of inequality, or "index of concentration," as he prefers to call it, such that, the greater δ, the greater inequality. This formula is a more convenient variant of Professor Pareto's, such that $\delta = \frac{a}{a-1}$, and, as a diminishes from any quantity greater than one down to one, δ increases up to infinity.

The equation $\log n = \delta \log s - \log c$ is easily transformed into that of a Lorenz curve. For, if N is the total number of income-receivers and S the total income, we have

$$\log N = \delta \log S - \log c.$$

$$\therefore \log \frac{n}{N} = \delta \log \frac{s}{S}.$$

Putting $\frac{n}{N} = \frac{y}{100}$ and $\frac{s}{S} = \frac{x}{100}$, we have the equation of a Lorenz curve,

$$\log \frac{y}{100} = \delta \log \frac{x}{100}$$

or

$$\frac{y}{100} = \left(\frac{x}{100}\right).$$

[1] See, *e.g.*, Pigou, *Wealth and Welfare*, pp. 25 and 72.

[2] See Bresciani, *Giornale degli Economisti*, August 1905, pp. 117-8, and January 1907, pp. 27–8. Ricci, *L'Indice di Variabilità*, pp. 43–6, Gini, *Variabilità* p. 65 and pp. 70–1. Compare also Persons, *Quarterly Journal of Economics*, 1908–9, pp. 420–1, and Benini, *Principii di Statistica Metodologica*, p. 187. Professor Benini inverts Professor Pareto's measure, but apparently without realising that he has done so.

[3] *Ibid.*, pp. 72 ff.

The area enclosed between this Lorenz curve and the line of equal distribution is—

$$\tfrac{1}{2}(100)^2 - \int_{x=0}^{x=100} y\,dx$$

$$= \tfrac{1}{2}(100)^2 - \int_0^{100} 100 \frac{x^\delta}{(100)^\delta}\,dx$$

$$= (100)^2\left(\tfrac{1}{2} - \frac{1}{\delta+1}\right).$$

Thus, the greater δ, the larger is the above area, and the larger the relative mean difference.[1] There is thus some ground for believing, though I do not here definitely commit myself to the belief, that *the reciprocal* of Professor Pareto's measure is a mere variant of the relative mean difference, in the particular case, when distribution is approximately according to Pareto's law. In this particular case, then, Professor Pareto's measure would have no independent significance, and, in the more general case, when distribution may depart widely from Pareto's law, the measure has, of course, no general significance at all. It will, therefore, be provisionally set aside in this discussion.

12. Returning to the four measures set out in order of merit at the end of Section 10, this order is based on theoretical advantages. But account must also be taken of practical applicability to statistics. Both the relative mean deviation and the quartile measure are more easily applicable than either of their two rivals to perfect statistics, and applicable, with less risk of serious error, to imperfect statistics. As regards perfect, or nearly perfect, statistics, the advantage of the former pair over the latter relates only to laboriousness and not to accuracy, and is not, therefore, a matter of great importance. But, as regards markedly imperfect statistics, such as are actually available, the advantage relates to accuracy as well as to laboriousness and is, therefore, vital.

The provisional conclusion which suggests itself, is as follows. When statistics are so imperfect, that neither the relative standard deviation nor the relative mean difference can be applied with any expectation of reasonable accuracy, we must make shift with the relative mean deviation and the quartile measure. It is some palliation of the comparative insensitiveness

[1] This index δ has been used by several Italian writers in enquiries into distributions of income. See, *e.g.*, Savorgnan, *La Distribuzione dei Redditi nelle Provincie e nelle Grandi Città dell' Austria*, and Porru, *La Concentrazione della Ricchezza nelle Diverse Regioni d'Italia*

to transfers, which is a defect of both the latter measures, that each is sensitive to many possible transfers, to which the other is insensitive. If, therefore, both give the same result in any particular comparison, their evidence is to some extent corroborative.

If statistics are so far improved that the relative standard deviation and the relative mean difference are applicable, these are to be preferred to the two measures just mentioned. If a single measure is to be used, the relative mean difference is, perhaps, slightly preferable, owing to the graphical convenience of the Lorenz curve. Probably, however, it will be desirable, at any rate for some time to come, not to rely upon the evidence of a single measure, but upon the corroboration of several. Given perfect, or nearly perfect, statistics, it is worth while considering whether corroboration may not also be sought from the measure $\frac{\log x_a}{\log x_g}$, applied, for the sake of simplicity, to total incomes, and not to surplus incomes in excess of the requirements of "bare subsistence." For this measure passes our test of proportionate additions to incomes, which none of the other four survivors do. In most practical cases, no doubt, these five measures will give results pointing in the same direction, but in some cases they may not do so.

Meanwhile, the chief practical necessity is the improvement of existing statistical information, especially as regards the smaller incomes. This paper may be compared to an essay in a few of the principles of brickmaking. But, until a greater abundance of straw is forthcoming, these principles cannot be put to the test of practice.

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