Answer Key on Exercises on Alkire Foster method

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Part I: Paper-Based Exercise on AF Measures

Given the following matrix of distribution of three dimensions (income, years of education, BMI and access to clean water):

$$X = \begin{bmatrix} 6 & 3 & 18 & 1 \\ 8 & 4 & 20 & 1 \\ 12 & 6 & 17 & 0 \\ 20 & 8 & 16 & 1 \\ 5 & 3 & 16 & 0 \end{bmatrix}$$

$$z = \begin{bmatrix} 10 & 6 & 18.5 & 1 \end{bmatrix}$$

a) **Calculate M0, H and A using a cutoff value of k=2 and equal weights**

g0 is the deprivation matrix computed using the weak deprivation cut-off, z

$$g0 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

Who is poor with k=2?

g0(k) is the censored deprivation matrix, c(k) the censored vector of deprivation count, and c(k)/d the censored vector of deprivation share.

$$g0(k) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad c(k) = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \\ 4 \end{bmatrix} \quad c(k)/d = \begin{bmatrix} 3/4 \\ 2/4 \\ 2/4 \\ 0 \\ 4/4 \end{bmatrix}$$

Calculated as the mean of the matrix:

M0=μ(g0(k))=11/20=0.55

Also note that:

H=4/5 That is, 80% of the population is poor (deprived in 2 or more dimensions).

A=((3+2+2+4)/4)/4=11/16 On average the poor (those deprived in 2 or more dimensions) are deprived in 69% of the indicators.

Then we can verify that the adjusted headcount ratio M0 is:

M0=H*A=(4/5)*(11/16)=11/20

So the poor in this society experience 55% of the total possible deprivations the society could experience.

b) Which is the contribution of the group of the first three individuals to overall M0?

$$M(x,y;z)=\frac{n_x}{n}M(x;z)+\frac{n_y}{n}M(y;z)$$

So the contribution of each group is: $C(x)=\left((n_x/n)M(x;z)\right)/M(x,y;z)$

In this case, group 'x' is composed of the first three people, so it is as taking a submatrix of the whole $g^0(k)$ matrix:

$$g_x^0(k)=\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&1&1\end{bmatrix}$$

M0(x)=7/12, H=1 A=((3+2+2)/4)/3=7/12 (Note that because in this case H=1, then M0=A)

Contribution (x)=[(3/5)*(7/12)]/(11/20)=7/11

In words: the group of the first three people contribute with 64% of overall poverty.

Now let's verify with the second group:

$$g_y^0(k)=\begin{bmatrix}0&0&0&0\\1&1&1&1\end{bmatrix}$$

M0(y)=1/2, H=1/2 A=1 (Note that because in this case A=1, then M0=H).

Contribution(y)=[(2/5)*(1/2)]/(11/20)=4/11

The group of the other two people contributes with 36% of overall poverty.

Note that the contribution of the two groups adds up to 100%.

c) What is the censored headcount ratio in each indicator and what does it mean? How does it differ from the 'raw' censored headcounts?

The censored headcount ratio in each indicator is 3/5 for income, education and BMI and 2/5 for water. That means that 60% of the population is poor and deprived in income, education and BMI, whereas 40% is poor and deprived in water. In this particular case, the censored headcount differ from the raw headcount in the case of the BMI indicator (where the raw headcount is 4/5, that is 80% of the population has low BMI).

d) Which is the contribution of each dimension to M0?

The contribution of each dimension to overall M_α is given by:

$$\text{Contrib}_j = \left(\mu(g^\alpha_{\bullet j}(k))/d\right)/M_\alpha(y;z) = \sum_{i=1}^n g^\alpha_{\bullet j}(k) \Big/ \sum_{i=1}^n \sum_{j=1}^d g^\alpha_{ij}(k)$$

In words: it is the proportion of people that have been identified as poor and are deprived in that particular dimension, divided by the total number of dimensions, and that as a proportion of overall poverty.

Income contribution=Education contribution=BMI contribution=

=((3/5)/4)/(11/20)=3/11

Water contribution=((2/5)/4)/(11/20)=2/11

If we add up all the contributions=3*(3/11)+2/11=1

Deprivation in income, education and BMI, each contribute with 27% of overall poverty whereas deprivation in water contributes with 18% of overall poverty.

e) What happens to H, M0, M1 and M2 if individual 1 becomes deprived in water? What happens to each of the mentioned measures if individual 1 experiences an income loss?

If individual 1 becomes deprived in water H does not change but M0, M1 and M2 will increase, because they satisfy *dimensional monotonicity*. Note that M0 will change here as this changes results in an additional deprivation for individual 1, which results in changes to A.

If individual 1 experiences an income loss H and M0 do not change, because they do not satisfy monotonicity. However, M1 and M2 will increase, because they satisfy monotonicity. Note that individual 1 is already deprived in income, and neither H nor M0 can account for the depth of deprivation that an individual experiences.

f) Re-do a-c using ranking weights: assigning a weight of 1.5 to income and education and 0.5 to water and BMI.

With these weights, the g0 matrix is now:

$$g0 = \begin{bmatrix} 1.5 & 1.5 & 0.5 & 0 \\ 1.5 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0 \\ 1.5 & 1.5 & 0.5 & 0.5 \end{bmatrix} \quad c = \begin{bmatrix} 3.5 \\ 3 \\ 1 \\ 0.5 \\ 4 \end{bmatrix}$$

Who is poor now with k=2?

$$g^0(k)=\begin{bmatrix}1.5 & 1.5 & 0.5 & 0\\ 1.5 & 1.5 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1.5 & 1.5 & 0.5 & 0.5\end{bmatrix} \quad c(k)=\begin{bmatrix}3.5\\ 3\\ 0\\ 0\\ 4\end{bmatrix}$$

Note that now the third person is not considered poor because the weights of BMI and water add up to less than 2. That is, the weighting structure affects both identification and aggregation.

M0=10.5/20=0.525

H=3/5 Now 60% are identified as poor.

A=((3.5+3+4)/4)/3=10.5/12 On average they are deprived in 87.5% of the weighted dimensions.

M0=H*A=(3/5)*(10.5/12)=10.5/20 So the poor are deprived in 52.5% of all the potential deprivations the society could experience.

Contribution of income=Contribution of education= 4.5/10.5

Contribution of BMI=1/10.5

Contribution of water=0.5/10.5

Contribution of the group of first three people=

=[(3/5)*(6.5/12)]/(10.5/20)=6.5/10.5 The first group contributes with 62% of total poverty.

We can verify that the second group contributes with 38%:

Contribution of the two remaining people=[(2/5)*(4/8)]/(10.5/20)=4/10.5

g) Same exercise using M1, k=2 and equal weights:

$$X=\begin{bmatrix}6 & 3 & 18 & 1\\ 8 & 4 & 20 & 1\\ 12 & 6 & 17 & 0\\ 20 & 8 & 16 & 1\\ 5 & 3 & 16 & 0\end{bmatrix}$$

$$z=\begin{bmatrix}10 & 6 & 18.5 & 1\end{bmatrix}$$

Based on the $g^0(k)$ matrix defined in a), we now create the $g^1(k)$ matrix (using k=2). Recall that each gap is given by: $g_{ij}^1(k)=(z_j-x_{ij})/z_j$ if $c_i>=k$ and $x_{ij}<z_j$, and 0 otherwise. In other words, $g_{ij}^1(k)=MAX((z_j-x_{ij})/z_j, 0)$.

$$g^1(k) = \begin{bmatrix} 0.4 & 0.5 & 0.027 & 0 \\ 0.2 & 0.33 & 0 & 0 \\ 0 & 0 & 0.081 & 1 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0.135 & 1 \end{bmatrix}$$

M1=0.234

We can verify that M1=H*A*G

H=4/5

A=11/16

G=(0.4+0.2+3*0.5+0.33+0.027+0.081+0.135+2)/11=0.425

M1=H*A*G=0.8*0.69*0.425=0.234

M1 can be interpreted as the average deprivation gap that the poor experience from the total potential deprivations that the society could experience. Note that M1 combines the headcount ratio, the intensity of poverty, and the average deprivation gap.

Contribution of income=[((0.4+0.2+0.5)/5)/4]/0.234=0.24

Contribution of education=[((0.5+0.33+0.5)/5)/4]/0.234=0.29

Contribution of BMI=[(0.027+0.081+0.135)/5)/4]/0.234=0.05

Contribution of water=[(2/5)/4]/0.234=0.43

We see that whenever we have an ordinal variable dichotomized, the gap is either 0 or 1 and therefore it implicitly receives a higher weight in the measure. That's why its contribution is so much higher. One should not use measures based on gaps, which require cardinal variables when one has dichotomous or ordinal variables.

Contribution of the first group of three people=54%

Contribution of the second group=46%

Using M2, k=2 and equal weights

Based on the $g^0(k)$ matrix defined in a), we now create the $g^2(k)$ matrix (using k=2). Recall that each gap is given by: $g_{ij}^2(k)=[(z_j-x_{ij})/zj]^2$ if $c_i>=k$ and $x_{ij}<z_j$, and 0 otherwise. In other words, $g_{ij}^1(k)=(MAX((z_j-x_{ij})/z_j, 0))^2$.

$$g^2(k) = \begin{bmatrix} 0.16 & 0.25 & 0.0007 & 0 \\ 0.04 & 0.111 & 0 & 0 \\ 0 & 0 & 0.007 & 1 \\ 0 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.0183 & 1 \end{bmatrix}$$

M2=0.154

We can verify that: M2=H*A*S

H=4/5

A=11/16

S=[0.16+0.04+3*0.25+0.111+2*0.0007+0.0183+2]/11=0.281

M2=(4/5)*(11/16)*0.281=0.154

M2 can be interpreted as the average squared deprivation gap that the poor experience from the total possible deprivations the society could experience. Note that it combines the headcount ratio, the intensity and the average squared deprivation gap. This last component makes this measure to be sensitive to a more equal distribution of poverty among the poor.

Contribution of income: [((0.16+0.04+0.25)/5)/4)]/0.154=0.146

Contribution of education: [((0.25+0.111+0.25)/5)/4]/0.154=0.198

Contribution of BMI: [((0.007+0.0007+0.0183)/5)/4]/0.154=0.008

Contribution of water: [(2/5)/4]/0.154=0.648

Again we see that deprivation in water has the 'highest contribution' but this is because it is a dichotomous variable.

Contribution of the group of the first three people=

(3/5)*[(0.16+0.25+0.0007+0.04+0.111+0.007+1)/12]/0.154=0.51

Contribution of the group of the last two people=

(2/5)*[(0.25+0.25+0.0183+1)/8]/0.154=0.49

If one wanted to calculate M1 or M2 using ranking weights one simply needs to multiply the gaps or the squared gaps by the corresponding weights and proceed in the same way as before.

For example, assume that we use the ranking weights of (e) and we keep k=2. Then the weighted g^1(k) and g^2(k) matrices are:

$$g^1(k)=\begin{bmatrix} 1.5*0.4 & 1.5*0.5 & 0.5*0.027 & 0 \\ 1.5*0.2 & 1.5*0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.5*0.5 & 1.5*0.5 & 0.5*0.135 & 0.5*1 \end{bmatrix}$$

M1=0.2115
Contribution of income=39%
Contribution of education 47%
Contribution of BMI=2%
Contribution of Water=12%
Contribution of the first group=51%
Contribution of the second group=49%

$$g^2(k)=\begin{bmatrix} 1.5*0.16 & 1.5*0.25 & 0.5*0.0007 & 0 \\ 1.5*0.04 & 1.5*0.111 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.5*0.25 & 1.5*0.25 & 0.5*0.0183 & 0.5*1 \end{bmatrix}$$

M2=0.105
Contribution of income=32.1%
Contribution of education=43.6%
Contribution of BMI=0.45%
Contribution of water=23.8%
Contribution of first group=40%
Contribution of second group=60%